Hardy's theorem and the prime number theorem

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June 2007

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Contents

Abstract

This BSc thesis has two main results. The first is Hardy's theorem (Theorem 17) about the zeros of the Riemannzeta function on the critical line. The second is the prime number theorem (Theorem 26) about the distribution of the prime numbers. Chapter 1 serves as a prelude to Chapter 2 and introduces the gamma function and extends it holomorphically to all of $\mathbb{C}$. Chapter 2 studies the Riemann zeta function, discusses the Riemann hypothesis, proves Hardy's theorem and introduces Dirichlet series. Chapter 3 states and proves the prime number theorem using Newman's proof, and Chapter 4 concludes and puts the thesis into perspective.

Resumé på dansk

Dette bachelorprojekt indeholder to hovedresulter. Det første er Hardys sætning (Sætning 17) om Riemann zeta-funktionens nulpunkter på den kritiske linje, mens det andet hovedresultat er primtalssætningen (Sætning 26) om fordelingen af primtal. Kapitel 1 tjener et indledende formål, idet det introducerer gammafunktionen og udvider den holomorft til hele $\mathbb{C}$. I kapitel 2 studeres Riemann zeta-funktionen, Riemann hypotesen diskuteres og Hardys sætning bevises. Til slut indføres Dirichletrækker. Kapitel 3 formulerer og beviser primtalssætningen ved hjælp af D.J. Newmans bevis, mens kapitel 4 konkluderer og perspektiverer projektet.

Jeg har valgt at skrive på engelsk for at kunne prøve at formulere en større tekst på engelsk, samt for at øge antallet af mulige læsere; heriblandt udenlandske venner og familie.

Auther's note—November 2007

After having received my grade my adviser Christian Berg has kindly provided me with all his corrections. This is not the original version of the thesis, but a version having adopted these corrections. Besides a ton of sign-errors[1] one serious mistake was made in the proof of Theorem 10 when finding and using the integrable majorant. The original version is available upon request.

[1] I tried so hard to avoid them, but now I truely understand why they are a mathematician's nightmare... Especially when two of them cancel each other out!

Preface

Writing the Bachelor of Science thesis is the first opportunity for an undergraduate student of mathematics at the University of Copenhagen to write an extensive mathematical text. Writing this has been a great learning experience in many ways. Being forced to write a mathematical text crystallized and emphasized much more clearly the ideas and thoughts of the proofs. I hope that this deeper level of understanding will benefit the reader. Another thrilling experience has been the additional creativity involving in writing compared with taking a course. I constantly question what material to include and what to leave out, and the Internet's endless generalizations and perspectives provide easy inspiration for further investigation.

A few people have contributed greatly to making my writing experience such a pleasant one, especially my very patient adviser, Christian Berg. He has given me considerable freedom in deciding how to structure this thesis and has listened intensively to my suggestions. During our many meetings, he has always taken every question of mine seriously, and it has been a great pleasure to experience his way of thinking about mathematics first hand.

Anders la Cour Bentzon, David Breuer and Morten Hornbech have all put in long hours and paid strict attention to detail while editing and packaging this thesis, providing myriad suggestions for rephrasing, changing the layout and correcting the ubiquitous spelling errors. Their work will make reading this thesis much more enjoyable.

The reader should be thoroughly familiar with basic and complex analysis to optimize understanding. Measure theory also plays an important role, but few of the arguments require more than Lesbegue's theorem. I believe that the beauty of Hardy's theorem and the prime number theorem lies in the fact that brilliant ideas and simple mathematics can prove deep theorems. I have therefore tried to keep things as simple as possible, which includes adding three appendixes with the results used throughout the thesis.

Jerôme Baltzersen
June 2007

CHAPTER 1

The gamma function, Bernoulli numbers and Bernoulli polynomials

1.1 Motivation of the gamma function

A wide variety of well-known functions has been defined for integers, such as the sum of the first n integers, the binomial coefficients and the factorial function. Consider the function

$$f(n) = 1 + \ldots + n, \quad f : \mathbb{N} \to \mathbb{N},$$

The formula $f(n) = \frac{n(n+1)}{2}$ allows f to be extended to becoming a function from $\mathbb{R}$ into $\mathbb{R}$. The gamma function is an attempt to construct a similar extension of the factorial function from $\mathbb{C}$ to $\mathbb{C}$. This section introduces the gamma function, but not all the steps are justified. Rather, this section scratches the surface and gives the reader an idea of the mathematical methods used to define and extend the gamma function, as these are quite similar to what is proved in detail when the Riemann zeta function is examined. This section also states and/or proves several functional equations and states a theorem discovered by Danish mathematicians Harald Bohr (1887–1951) and Johannes Mollerup (1872–1937), that uniquely characterizes the gamma function among the class of functions that yield factorials at integer values. It is quite clear that the number of such functions is infinite. Roughly speaking, all that is required is to plot the factorials at integer values and then draw a graph through all of these. More rigorously, if f is any such function, then $g(x) = p(x)f(x)$, where p is any 1-periodic function with $p(1) = 1$, is another such function. The Bohr-Mollerup theorem is a surprisingly simple and beautiful characterization of the gamma function among this infinite class of functions and has powerful applications. Finally, this section extends the gamma function to complex variables. The gamma function plays an important role in the later examination of the Riemann-zeta function, but it also turns out to be ubiquitous in probability theory, physics and numerous other theories as well.

1.2 The gamma function

The first task is to define the gamma function for positive real numbers. Note that there are many equivalent definitions of the gamma function, but this thesis chooses the most popular one:

Definition 1 (The gamma function)**.** *The* **gamma function** $\Gamma : (0, \infty) \to (0, \infty)$ *is defined by the integral*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.1}$$

It is immediately apparent that $\Gamma(x)$ cannot be negative, since for all $x \in (0, \infty)$ the integrand is positive. This definition creates two potential problems: 1) what happens when t gets close to 0, and 2) the domain of integration is infinite. Hence consider

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt \leq \int_0^1 t^{x-1}dt + \int_1^\infty t^{x-1}e^{-t}dt < \infty,$$

where we know that the integral from 1 to ∞ is finite, as e^{-t} decreases much faster than any power of t, and that $\int_0^1 t^{x-1}dt < \infty$ if and only if $x > 0$. Integrating by parts, we find

$$\int_{1/n}^n t^x e^{-t}dt = -[t^x e^{-t}]_{1/n}^n + x\int_{1/n}^n t^{x-1}e^{-t}dt,$$

as $-[t^x e^{-t}]_{1/n}^n \to 0$ as $n \to \infty$ Lebesgue's monotone convergence theorem B.1, which gives us

$$\Gamma(x+1) = x\Gamma(x). \tag{1.2}$$

This functional equation is a strong characterization of the gamma function and a key component in the extension of the gamma function to $\mathbb{C}$, as this thesis shows later on. Combined with the fact that $\Gamma(1) = 1$, we get

$$\Gamma(n+1) = n!, \tag{1.3}$$

which shows the close connection with the factorial function. Note that the process of evaluating the gamma function for any positive real number is reduced to knowing the value of $\Gamma(x)$ for $0 < x < 1$. The following calculation for $\lambda \in (0,1)$ shows that $\Gamma(x)$ is log-convex:

$$\begin{aligned}\Gamma(\lambda a + (1-\lambda)b) &= \int_0^\infty t^{\lambda a+(1-\lambda)b-1}e^{-t}dt = \int_0^\infty (t^{a-1}e^{-t})^\lambda \cdot (t^{b-1}e^{-t})^{1-\lambda}dt \\ &\leq \left(\int_0^\infty t^{a-1}e^{-t}dt\right)^\lambda \cdot \left(\int_0^\infty t^{b-1}e^{-t}dt\right)^{1-\lambda} = \Gamma(a)^\lambda \cdot \Gamma(b)^{1-\lambda},\end{aligned}$$

using Hölder's theorem (B.3) to establish the inequality.

This gives us the three criteria needed to single out the gamma function.

Theorem 2 (Bohr-Mollerup)**.** *Let f be any function satisfying the following three conditions for $x > 0$:*

1. *$f(x+1) = xf(x)$.*
2. *f is log-convex.*
3. *$f(1) = 1$.*

Then $f(x) = \Gamma(x)$ for $x > 0$.

Proof. It has been shown that the gamma function actually satisfies the three conditions. The proof is omitted and can be found in Artin [5]. □

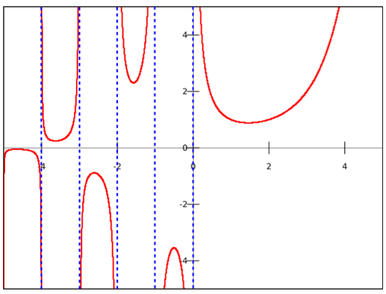

Figure 1.1: Plot of the gamma function along part of the real axis.

Numerous formulae involve the gamma function, and this thesis only mentions a few. Every so often the gamma function is referred to as *Euler's second integral*, the first being

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y>0, \tag{1.4}$$

which is usually called the *beta function*. It is very surprising that the beta and the gamma function are strongly related:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.5}$$

We wish to show this relation using the Bohr-Mollerup theorem. First observe that

$$\begin{aligned} B(x+1,y) = \int_0^1 \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}\,dt = -\left[\frac{(1-t)^{x+y}}{x+y}\left(\frac{t}{1-t}\right)^x\right]_0^1 \\ + \int_0^1 \frac{x}{x+y}(1-t)^{x+y}\left(\frac{t}{1-t}\right)^{x-1}\frac{1}{(1-t)^2}\,dt = \frac{x}{x+y}B(x,y), \end{aligned} \tag{1.6}$$

where we have used integration by parts. It now follows that $f(x) = B(x,y)\cdot\Gamma(x+y)$ fulfills $f(x+1) = xf(x)$: that is, condition 1 of the Bohr-Mollerup theorem. Equation (A.3) shows that the beta function is log-convex, and the product of two log-convex functions is again log-convex (cf. Appendix A). Further,

$$B(1,y) = \int_0^1 (1-t)^{y-1}dt = \frac{1}{y}, \quad \text{and therefore} \quad f(1) = \frac{1}{y}\Gamma(1+y) = \Gamma(y),$$

but then $g(x) = \frac{f(x)}{\Gamma(y)}$ must fulfill all three requirements of the Bohr-Mollerup theorem, and hence g is the gamma function. Inserting the definition of f and rearranging, provides formula (1.5). Euler also proved a reflection formula, known as Euler's formula:

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}, \quad x \notin \mathbb{Z}. \tag{1.7}$$

The duplication formula found by Legendre is

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x+1/2)\Gamma(x). \tag{1.8}$$

From (1.7) as well as (1.8) it is easy seen that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Gauss later generalized this to

$$\prod_{i=0}^{p-1}\Gamma\left(\frac{x+i}{p}\right) = \Gamma\left(\frac{x}{p}\right)\cdots\Gamma\left(\frac{x+p-1}{p}\right) = \frac{(2\pi)^{p-1/2}}{p^{x-1/2}}\Gamma(x),$$

known as the multiplication formula. A last formula involving the gamma function, which this thesis uses later, is the following interesting interplay with the trigonometric functions

$$\begin{aligned} &\int_0^\infty t^{x-1}\cos(bt)dt = \cos\left(\frac{\pi}{2}x\right)b^{-x}\Gamma(x) \\ &\int_0^\infty t^{x-1}\sin(bt)dt = \sin\left(\frac{\pi}{2}x\right)b^{-x}\Gamma(x), \quad b>0, \quad 0<x<1. \end{aligned} \tag{1.9}$$

As a special case with $b = 1$ and $x := -x$, we get

$$\int_0^\infty t^{-x-1}\sin t\,dt = -\sin\left(\frac{\pi x}{2}\right)\Gamma(-x). \tag{1.10}$$

All these formulae can be extended to complex arguments, although caution has to be exercised in (1.9), where $0 < \Re(z) < 1$.

1.3 Bernoulli numbers and Bernoulli polynomials

From this point on, complex analysis plays an important role in the discussion. A brief appendix includes some important theorems, but textbooks on this subject thoroughly treat complex analysis, such as Berg [1]. This section defines the Bernoulli numbers and polynomials, which will show up again and again in the quest for proving the prime number theorem. The Bernoulli numbers were first discovered in the study of Faulhaber's formula, which is described in [10]. Note that the definition of Faulhaber's formula is motivated entirely in algebra and number theory, whereas this thesis defines the Bernoulli numbers using complex analysis. This is the first time in this thesis that simple analysis is applied to a number theoretical problem.

Consider the function

$$f(z) = \frac{z}{e^z - 1}, \quad z \in B(0, 2\pi).$$

It is a complex valued function holomorphic in $B(0, 2\pi)$ with a removable singularity in $z = 0$. Using L'Hospital's rule, we find that $\lim_{z\to 0} \frac{z}{e^z-1} = 1$.

Definition 3 (Bernoulli numbers)**.** *The* Bernoulli numbers $B_0, B_1, \ldots$ *are defined as the coefficients in the Taylor expansion of f in $B(0, 2\pi)$:*

$$f(z) = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \quad z \in B(0, 2\pi). \tag{1.11}$$

Comparing Taylor-expansions shows some basic properties of the Bernoulli numbers. For $z \in B(0, 2\pi)$, consider

$$z = (e^z - 1)\frac{z}{e^z - 1} = \left(\sum_{k=1}^{\infty} \frac{1}{k!} z^k\right)\left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k\right) = B_0 z + \sum_{n=2}^{\infty}\left(\sum_{k=0}^{n-1} \frac{B_k}{k!} \cdot \frac{1}{(n-k)!}\right) z^n,$$

from which it follows that $B_0 = 1$, and

$$\sum_{k=0}^{n-1} \frac{1}{k!(n-k)!} B_k = 0 \implies \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \text{ for } n \geq 2. \tag{1.12}$$

This demonstrates that the Bernoulli numbers are rational, and we further now have an algorithm to calculate the nth Bernoulli number. For instance, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$. Now consider the function $\phi(z) = \frac{z}{e^z-1} - 1 + \frac{z}{2} = \sum_{k=2}^{\infty} \frac{B_k}{k!} z^k$.

$$\phi(-z) = \frac{-ze^z}{1 - e^z} - 1 - \frac{z}{2} = \frac{-ze^z - z(1 - e^z)}{1 - e^z} - 1 + \frac{z}{2} = \frac{-z}{1 - e^z} - 1 + \frac{z}{2} = \phi(z),$$

which shows that ϕ is an even function. Hence we see that the odd coefficients in its Taylor series must all be 0, which means that the odd Bernoulli numbers, except B_1, all equal 0. We now introduce the Bernoulli polynomials.

Definition 4 (Bernoulli polynomials)**.** *The* Bernoulli polynomials $B_0(t), B_1(t), \ldots$ *are defined as $B_0(t) = 1$, and for $n \geq 1$ by the recursion formula*

$$B_n'(t) = nB_{n-1}(t) \quad \textit{and} \quad \int_0^1 B_n(t)dt = 0. \tag{1.13}$$

A priori the nth Bernoulli polynomial has degree n, where the integral in (1.13) determines the constant. Observation: this implies that the Bernoulli polynomials are uniquely defined. The following theorem more completely characterizes this.

Theorem 5. *The Bernoulli polynomials fulfill the following:*

(i) The nth Bernoulli polynomial is given by

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, t^k \tag{1.14}$$

(ii) The following symmetry property holds:

$$B_n(1-t) = (-1)^n B_n(t). \tag{1.15}$$

(iii) For all n, $B_n(0) = B_n$, and for $n \geq 2$, $B_n(0) = B_n(1) = B_n$.

Proof. First note that (iii) follows trivially from (i) and (ii), as the odd Bernoulli numbers except B_1 all equals 0. To show (i) and (ii), we will use the same strategy. First, we shall consider a sequence of polynomials $P_n(t)$ and then show that they satisfy (1.13) and $P_0(t) = 1$: that is, they are the Bernoulli polynomials according to the previous observation on the uniqueness in the definition.

(i) Hence consider $P_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} t^k$. It is clear that $P_0(t) = 1$, as $\binom{0}{0} = 1$.

$$\begin{aligned} P_n(t)' &= \sum_{k=1}^{n} k \cdot \binom{n}{k} B_{n-k}\, t^{k-1} = \sum_{k=1}^{n} k \frac{n(n-1)!}{k(k-1)!\big((n-1)-(k-1)\big)!} B_{n-k}\, t^{k-1} \\ &= \sum_{k=1}^{n} n \binom{n-1}{k-1} B_{(n-1)-(k-1)}\, t^{k-1} = n \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}\, t^k = nP_{n-1}(t). \end{aligned}$$

We now want to show the second part of definition 4: that $\int_0^1 P_n(t)dt = 0$.

$$n \int_0^1 P_{n-1}(t)dt = \int_0^1 P_n'(t)dt = P(1) - P(0) = \sum_{k=1}^{n} \binom{n}{k} B_{n-k} = \sum_{j=0}^{n-1} \binom{n}{j} B_j,$$

because $P(0) = B_n$, using the substitution $j = n - k$ and due to the identity $\binom{n}{k} = \binom{n}{n-k}$. Because of (1.12), which holds for $n \geq 2$, it now follows

$$n \int_0^1 P_{n-1}(t)dt = \sum_{j=0}^{n-1} \binom{n}{j} B_j = 0,$$

and thus $\int_0^1 P_n(t)dt = 0$ for $n \geq 1$. Therefore, $P_n(t) = B_n(t)$ for all n and $t \in [0,1]$.

(ii) This proof follows the lines of the above. Assume $Q_n(t) = (-1)^n B_n(1-t)$. Note that $Q_n'(t) = (-1)^{n-1}\, n\, B_{n-1}(1-t) = n\, Q_{n-1}(t)$, and that, by substitution,

$$\int_0^1 Q_n(t)dt = (-1)^{n+1} \int_1^0 B_n(t)dt = 0.$$

As $Q_0(t) = 1$ as well, $Q_n(t)$ must be the Bernoulli polynomials. □

The Bernoulli numbers and polynomials play an important role considering the Riemann zeta function. The following identity for $t \in [0,1[$ enables specific values of the Riemann zeta function to be calculated:

$$\begin{aligned} B_{2n-1}(t) =& (2n-1)!(-1)^n \cdot 2 \sum_{k=1}^{\infty} \frac{\sin(2\pi k t)}{(2\pi k)^{2n-1}} \\ B_{2n}(t) =& (2n)!(-1)^{n-1} \cdot 2 \sum_{k=1}^{\infty} \frac{\cos(2\pi k t)}{(2\pi k)^{2n}}, \end{aligned} \tag{1.16}$$

for $n \geq 1$ and $k \in \mathbb{N}$, and is proved in §6 of Berg [2] using Fourier series. We also need Table 1.1 as we move on.

$B_0(t)$	$B_1(t)$	$B_2(t)$	$B_3(t)$	$B_4(t)$	$B_5(t)$
0	$t - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$

Table 1.1: The first few Bernoulli polynomials

1.4 The extension of the gamma function to complex argument

We have previously considered the gamma function for positive real numbers, and this section extends the gamma function to complex argument. It is extended to a meromorphic function in $\mathbb{C}$ with simple poles at $\{0, -1, -2, ...\}$. The fundamental functional equation $\Gamma(x+1) = x\Gamma(x)$ (1.2) will be shown to hold for complex values as well (1.17), and as we shall see, it will play a decisive role in the extension. But first let us consider the half plane $\mathbb{H} := \{z \in \mathbb{C} | \Re(z) > 0\}$.

For $x + iy = z \in \mathbb{H}$, $|t^{z-1}e^{-t}| \leq |t^{x-1}e^{-t}|$, as $|t^z| = |t^x e^{i\log(t)y}| = |t^x|$. The gamma function is therefore well defined in $\mathbb{H}$, as

$$|\Gamma(z)| := \left| \int_0^\infty t^{z-1}e^{-t}dt \right| \leq \int_0^\infty t^{x-1}e^{-t}dt = \Gamma(x) < \infty, \text{ for } z \in \mathbb{H}.$$

To show that $\Gamma(z)$ is holomorphic, we introduce the auxiliary function $\Gamma_n(z)$, defined as

$$\Gamma_n(z) = \int_{\frac{1}{n}}^n t^{z-1}e^{-t}dt = \int_{\frac{1}{n}}^n e^{(z-1)\log(t)-t}dt$$

which is well defined in the entire complex plane and, based on Theorem (C.2) actually holomorphic as well. We will proceed to show that $\Gamma_n(z) \to \Gamma(z)$ locally uniformly for $z \in \mathbb{H}$, as this will allow us to conclude that Γ is holomorphic in $\mathbb{H}$ using Theorem C.1. First we introduce $S_{a,b} := \{z \in \mathbb{C} | a \leq \Re(z) \leq b\}$ for $0 < a < b$. $S_{a,b} \subset \mathbb{H}$, and for $z \in S_{a,b}$, we obtain

$$|\,\Gamma(z) - \Gamma_n(z)| \leq \left| \int_0^{\frac{1}{n}} t^{x-1}e^{-t}dt + \int_n^\infty t^{x-1}e^{-t}dt \right| \leq \left| \int_0^{\frac{1}{n}} t^{a-1}e^{-t}dt \right| + \left| \int_n^\infty t^{b-1}e^{-t}dt \right|,$$

which tends to 0 as n tends to ∞, and hence $\Gamma_n(z) \to \Gamma(z)$ locally uniformly for $z \in S_{a,b}$. As a and b were arbitrarily chosen, this conclusion holds for the whole of $\mathbb{H}$ on letting $a \to 0$ and $b \to \infty$. To show that

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{H}, \tag{1.17}$$

we only need to note that both sides are holomorphic functions in $\mathbb{H}$, and as we have previously shown that they agree on $]0, \infty[$, it follows from the identity theorem for holomorphic functions that they must agree on the whole of $\mathbb{H}$. The basic idea is now the following: Rearranging in (1.17), we get an expression for $\Gamma(z)$ given by $\Gamma(z+1)/z$, which is well defined for $\Re(z) > -1$ with a simple pole at $z = 0$. We can determine the residue at 0:

$$\text{Res}(\Gamma, 0) = \lim_{z \to 0}(z-0)\Gamma(z) = \lim_{z \to 0} z\frac{\Gamma(z+1)}{z} = \Gamma(1) = 1.$$

In this way we have added the strip $-1 < \Re(z) \leq 0$ to the domain of the gamma function. As we have used (1.17) to extend our function, this particular identity will hold for the extended domain. Continuing in this way, we can extend the gamma function one strip at a time, and we end up with a meromorphic function in all of $\mathbb{C}$ with simple poles at $\{0, -1, -2, ...\}$. We find the residues for $n \in \mathbb{N}$ to be:

$$\begin{aligned}\text{Res}(\Gamma, -n) &= \lim_{z \to -n}(z+n)\Gamma(z) = \lim_{z \to -n}(z+n)\frac{\Gamma(z+1)}{z} = \lim_{z \to -n}(z+n)\frac{\Gamma(z+2)}{z(z+1)} = \dots \\ &= \lim_{z \to -n}(z+n)\frac{\Gamma(z+n+1)}{z(z+1)\cdot \ldots \cdot (z+n)} = \frac{1}{(-1)^n \cdot n!}.\end{aligned}$$

Hence $\Gamma(z+1) = z\Gamma(z)$ holds for $z \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, and we have succeeded in extending the gamma function to the whole complex plane.

CHAPTER 2

The Riemann zeta function

This chapter defines the Riemann zeta function[1] denoted by $\zeta(z)$. The Bernoulli numbers play an important role in evaluating $\zeta(z)$ for particular values, and the functional equation for the zeta function is given by

$$\zeta(z) = \frac{1}{\pi}(2\pi)^z \Gamma(1-z)\zeta(1-z)\sin\left(\frac{\pi z}{2}\right),$$

which is important for a discussion of the Riemann hypothesis. The zeta function is a special case of Dirichlet series, which this thesis studies in detail later. The *Möbius function* $\mu(n)$, produces the following expression:

$$\frac{1}{\zeta(z)} = \sum_{n=0}^{\infty} \frac{\mu(n)}{n^z}, \quad \Re(z) > 1.$$

2.1 Definition and Euler's product

Definition 6 (zeta function)**.** *The* zeta function *on* $\{z \in \mathbb{C} | \Re(z) > 1\} \to \mathbb{C}$ *is a holomorphic function defined and is given by*

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{2.1}$$

It is clear that whenever $\Re(z) > 1$, $\zeta(z)$ is finite, and ζ is therefore well defined. Further, when $t = 0$ is inserted into (1.16), we get

$$B_{2n} = B_{2n}(0) = (2n)!(-1)^{n-1} \cdot 2\sum_{k=1}^{\infty} \frac{1}{(2\pi k)^{2n}} = (2n)!(-1)^{n-1} \cdot 2\frac{1}{(2\pi)^{2n}}\zeta(2n),$$

and rearranging,

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n}. \tag{2.2}$$

This allows us to evaluate $\zeta(z)$ for positive even integers, and by recalling that $B_2 = \frac{1}{6}$, it in particular provides one of many solutions to the Basel problem,[2]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[1] There are many other zeta functions, but as this thesis only uses the Riemann zeta function, we shall henceforth call it the "zeta function" and denote it by $\zeta(z)$.

[2] The problem of evaluating $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known as the Basel problem. Although Euler was the first to solve it, he did not do this by means of the zeta function. This is described in more detail in [11].

We now proceed to show Euler's product formula for $\zeta(z)$, which introduces prime numbers in this thesis for the first time.

Theorem 7 (Euler's product)**.** *For $\Re(z) > 1$, we have $\zeta(z) \neq 0$, and the following formula holds:*

$$\zeta(z) = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^z}}, \tag{2.3}$$

where $\mathbb{P}$ is the set of primes. The product converges uniformly in every half plane $\Re(z) \geq a$ for every $a > 1$.

Proof. Consider $\frac{1}{2^z}\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{(2n)^z}$, which implies that

$$\zeta(z)(1-\frac{1}{2^z}) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \frac{1}{11^z}\dots,$$

that is, all the terms of the sum with a denominator divisible by 2 have been removed. We can repeat the process for 3:

$$\zeta(z)(1-\frac{1}{2^z})(1-\frac{1}{3^z}) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z}\dots.$$

Continuing this for the first N primes, we have

$$\zeta(z)\prod_{n=1}^{N}(1-\frac{1}{p_n^z}) = 1 + \frac{1}{m_1^z} + \frac{1}{m_2^z} + \dots,$$

where $m_1, m_2, \dots$ are the first natural numbers not divisible by any of the N first primes, and hence we have that $\{m_1, m_2, \dots\} \subset \{p_{N+1}, p_{N+1}+1, \dots\}$. We now get for $z = x + iy$

$$\left|\zeta(z)\prod_{n=1}^{N}(1-\frac{1}{p^z}) - 1\right| \leq \frac{1}{m_1^x} + \frac{1}{m_2^x} + \dots \leq \sum_{k=P_N+1}^{\infty} \frac{1}{n^x}.$$

For any $x \geq a > 1$, this expression tends to 0 as $N \to \infty$, and hence the product from (2.3) converges uniformly in every half plane $\Re(z) \geq a > 1$. □

As an interesting consequence of the above theorem, it is possible to show that $\sum_{\mathbb{P}} \frac{1}{p}$ is divergent. First, we note that $\log\frac{1}{1-x} \leq 2x$ for $0 \leq x \leq \frac{1}{2}$. This estimation holds for every term in the product of (2.3), and as the logarithm transforms the product into a sum, we have:

$$\log\zeta(x) = \sum_{\mathbb{P}} \log\frac{1}{1-1/p^x} \leq 2\sum_{\mathbb{P}}\frac{1}{p^x}, \quad x > 1. \tag{2.4}$$

Letting $x \to 1^+$, we get the desired series, and using the result from (B.1), we find

$$\infty = \lim_{x\to1^+}\log\zeta(x) \leq 2\sum_{\mathbb{P}}\frac{1}{p}. \tag{2.5}$$

2.2 Meromorphic extension of the zeta function and its functional equation

Extending the zeta function requires proving quite a few results, the first being:

Theorem 8. *For $\Re(z) > 1$, the following holds:*

$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{t^{z-1}}{e^t - 1}\,dt.$$

Proof. Recalling the definition of the gamma function from (1.1) and using the substitution $u = nt$, we get

$$\Gamma(z) = \int_0^\infty n^{z-1}t^{z-1}e^{-nt} \cdot n\,dt = n^z \int_0^\infty \frac{t^{z-1}}{e^{nt}}\,dt.$$

As this holds for any $n \in \mathbb{N}$, we find that

$$\zeta(z)\Gamma(z) = \sum_{\mathbb{N}} \int_0^\infty \frac{t^{z-1}}{e^{nt}}\,dt = \int_0^\infty t^{z-1} \sum_{\mathbb{N}} e^{-nt}\,dt = \int_0^\infty t^{z-1} \frac{e^{-t}}{1-e^{-t}}\,dt = \int_0^\infty \frac{t^{z-1}}{e^t - 1}\,dt,$$

where we have used that $\sum_{\mathbb{N}} e^{-nt}$ is a geometric series, and where we can interchange summation and integration due to Lebesgue's Theorem B.2, since

$$\left| \sum_{\mathbb{N}} \frac{t^{z-1}}{e^{nt}} \right| \leq \frac{t^{x-1}}{e^t - 1}$$

is an integrable upper bound for $x > 1$. □

Let $\overline{B}_n(t) = B_n(t - \lfloor t \rfloor)$, which is called the nth periodic Bernoulli polynomial. Then the *Euler-Maclaurin sum formula*—which we shall not prove here—for a function $f \in C^1[0, n]$, states that

$$\frac{f(0) + f(n)}{2} + \sum_{k=1}^{n-1} f(k) - \int_0^n f(t)dt = \int_0^n \overline{B}_1(t) f'(t) dt.$$

Considering $f(t) = (t + 1)^{-z}$, we find for $\Re(z) > 1$

$$\frac{1 + (n+1)^{-z}}{2} + \sum_{k=2}^{n} \frac{1}{k^z} = \int_0^n (t+1)^{-z} dt + \int_0^n -z \frac{\overline{B}_1(t)}{(t+1)^{z+1}} dt.$$

Rearranging and adding 1 to both sides to obtain index $k = 1$ in the sum, we find

$$\sum_{k=1}^{n} \frac{1}{k^z} = \frac{1}{2} - \frac{(n+1)^{-z}}{2} + \int_1^{n+1} t^{-z} dt - z \int_0^n \frac{\overline{B}_1(t)}{(t+1)^{z+1}} dt. \tag{2.6}$$

As $\Re(z) > 1$ for $n \to \infty$, we find that $\int_1^{n+1} t^{-z} dt = [\frac{-1}{z-1} t^{-z+1}]_1^{n+1} \to \frac{1}{z-1}$, and hence the above equation (2.6) tends to

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} - z \int_0^\infty \frac{\overline{B}_1(t)}{(t+1)^{z+1}} dt. \tag{2.7}$$

Now consider the function $\phi(z) = \int_0^\infty \frac{\overline{B}_1(t)}{(t+1)^{z+1}} dt = \int_1^\infty \frac{\overline{B}_1(t)}{t^{z+1}} dt$, as $\overline{B}_1(t)$ has period 1. We want to show that ϕ is holomorphic in $\mathbb{H}$, and once again we use Theorem C.1 as we did when we extended the gamma function. Hence, we need to show that

$$\phi_p(z) = \int_1^p \frac{\overline{B}_1(t)}{t^{z+1}} dt \to \int_1^\infty \frac{\overline{B}_1(t)}{t^{z+1}} dt$$

uniformly for $\Re(z) \geq a > 0$ as $p \to \infty$. Note that $|\overline{B}_1(t)| \leq \frac{1}{2}$, and hence we have

$$|\phi(z) - \phi_p(z)| = \left| \int_p^\infty \frac{\overline{B}_1(t)}{t^{z+1}} dt \right| \leq \int_p^\infty \frac{1/2}{t^{a+1}} dt = \frac{1}{2ap^a} \to 0 \quad \text{for} \quad p \to \infty.$$

We now proceed to show that $\phi_p(z)$ converges uniformly for compact sets K in $\Re(z) > -1$. First, note that for such sets $1_{(1,\infty)}(t) \frac{\overline{B}_1(t)}{t^{z+1}}$ is not necessarily integrable, and we shall therefore consider the improper integral as the limit of

$$\phi_q(z) = \int_1^q \frac{\overline{B}_1(t)}{t^{z+1}} dt, \quad q \to \infty.$$

To show uniform convergence, we then need to show $\left|\int_p^q \frac{\overline{B}_1(t)}{t^{z+1}}dt\right| \to 0$ for $p, q \to \infty$. First we note, integrating by parts as $\overline{B}_2'(t) = 2\overline{B}_1(t)$ by (1.13), that

$$\int_p^q \frac{\overline{B}_1(t)}{t^{z+1}}\,dt = \left[\frac{\overline{B}_2(t)}{2t^{z+1}}\right]_p^q + (z+1)\int_p^q \frac{\overline{B}_2(t)}{2t^{z+2}}\,dt, \tag{2.8}$$

as $\Re(z) \geq a > -1$. As K is compact, we can find a constant A such that $|z| \leq A$. Recall from Table 1.1 that $|\overline{B}_2(t)| \leq 1/6$, and thus

$$\left|\left[\frac{\overline{B}_2(t)}{2t^{z+1}}\right]_p^q\right| \leq \frac{1}{12}\left|\left(\frac{1}{q^{z+1}} - \frac{1}{p^{z+1}}\right)\right| \leq \frac{1}{12}\left(\frac{1}{q^{a+1}} + \frac{1}{p^{a+1}}\right),$$

$$\left|(z+1)\int_p^q \frac{\overline{B}_2(t)}{2t^{z+2}}dt\right| \leq (A+1)\frac{1}{12}\left|\left[\frac{-1}{z+1}\frac{1}{t^{z+1}}\right]_p^q\right| \leq \frac{A+1}{12(a+1)}\left(\frac{1}{q^{a+1}} + \frac{1}{p^{a+1}}\right),$$

from which it is apparent that both terms in (2.8) tend to 0 as $p, q \to \infty$, and we have therefore shown that the improper integral $\phi_p(z)$ converges uniformly on compact sets K. Using Theorem C.1 the improper integral defines a holomorphic function of z for $\Re(z) > -1$, and we can therefore obtain a meromorphic extension of the zeta function given by

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} - z\int_1^\infty \frac{\overline{B}_1(t)}{t^{z+1}}dt, \quad \Re(z) > -1 \tag{2.9}$$

with a simple pole at $z = 1$. The generalization of this extension is obvious. For each partial integration we increase the power of the denominator under the integral sign by 1, and during this process we add another strip to the set of the definition. Inductively, we extend the zeta function to the whole of $\mathbb{C}$. To illustrate, we integrate by parts once more and find:

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} + \frac{zB_2}{2} - z(z+1)\int_1^\infty \frac{\overline{B}_2(t)}{2t^{z+2}}dt, \quad \Re(z) > -2$$

Note that, for each step, we can evaluate the zeta function for yet another non-positive integer. For instance, by the two expansions above, $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{2}B_2 = -\frac{1}{12}$. This, however, is of limited interest, as we shall prove a functional equation for the zeta function that yields these results more easily. In conclusion, we have extended the zeta function to all of $\mathbb{C}$ with a simple pole at $z = 1$.

As a curiosity, this provides a way to interpret the infinite divergent sums of $1 + 1 + \ldots$ and $1 + 2 + 3 + \ldots$ as $-\frac{1}{2}$ and $-\frac{1}{12}$, respectively. As absurd as this may seem, these results are used in the physical theories of quantum mechanics and string theory; cf. [12].

Before we turn to prove the functional equation for the zeta function (Theorem 10), we will mention a lemma and remark the following:

$$\zeta(z) = -z\int_0^\infty \frac{\overline{B}_1(t)}{t^{z+1}}dt, \quad -1 < \Re(z) < 0. \tag{2.10}$$

Why? According to (2.9), we will have to show that for $-1 < \Re(z) < 0$,

$$\frac{1}{2} + \frac{1}{z-1} = -z\int_0^1 \frac{\overline{B}_1(t)}{t^{z+1}}dt = -z\int_0^1 \frac{1}{t^z}dt + \frac{z}{2}\int_0^1 \frac{1}{t^{z+1}}dt = -z\left[\frac{t^{1-z}}{1-z}\right]_0^1 + \frac{z}{2}\left[\frac{t^{-z}}{-z}\right]_0^1,$$

remembering that $B_1(t) = t - \frac{1}{2}$. The above integrals are well defined as $-1 < \Re(z) < 0$, and the following calculations are meaningful for the same reason:

$$-z\left[\frac{t^{1-z}}{1-z}\right]_0^1 + \frac{z}{2}\left[\frac{t^{-z}}{-z}\right]_0^1 = \frac{z}{z-1} - \frac{1}{2} = \frac{z-1+1}{z-1} - \frac{1}{2} = \frac{1}{2} + \frac{1}{z-1}.$$

Lemma 9. *For $\theta \in \mathbb{R}$ and any $n \in \mathbb{N}$,*

$$\left|\sum_{k=1}^n \frac{\sin k\theta}{k}\right| \leq \pi + 2. \tag{2.11}$$

Proof. The proof is omitted and can be found in Berg [2]. □

We are now well suited to proceed to show the functional equation.

Theorem 10 (functional equation for the zeta function)**.** *The zeta function satisfies the following functional equation:*

$$\zeta(z) = \frac{1}{\pi}(2\pi)^z \Gamma(1-z)\zeta(1-z)\sin\frac{\pi z}{2}, \tag{2.12}$$

or, equivalently,

$$\zeta(z)\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}} = \zeta(1-z)\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}} \tag{2.13}$$

for all $z \in \mathbb{C}\setminus\{0,1\}$*. The latter formula is noted to be symmetric with respect to* $z = \frac{1}{2}$*.*

Proof. We begin by recalling that $\zeta(z)$ is holomorphic in $\mathbb{C}\setminus\{1\}$. Looking at the right side of (2.12), we observe that $\zeta(1-z)$ is holomorphic in $\mathbb{C}\setminus\{0\}$ while $\Gamma(1-z)$ is holomorphic $\mathbb{C}\setminus\{1,2,\ldots\}$. The right side is therefore holomorphic in $\mathbb{C}\setminus\{0,1,2,\ldots\}$, and according to the identity theorem for holomorphic functions, (2.12) holds if we can show the equation for z with $-1 < \Re(z) < 0$. Before this is done, we analyze (2.12) for $z \in \{0,2,3,\ldots\}$.

Since $\zeta(z)$ is holomorphic in $\mathbb{C}\setminus\{1\}$, the right side must have removable singularities at $z \in \{0,2,3,\ldots\}$. From (2.9), we have previously concluded that $\zeta(z)$ has a simple pole at $z=1$, and hence $\zeta(1-z)$ has one at $z=0$. Further, $\sin\frac{\pi z}{2}$ has a simple zero at $z=0$. This simple pole and the simple zero "cancels" each other: Let g be such that $\sin\frac{\pi z}{2} = zg(z)$ with $g(0)\neq 0$. Then

$$\lim_{z\to 0}\zeta(1-z)\sin\frac{\pi z}{2} = \lim_{z\to 0} z\zeta(1-z)g(z),$$

which is a well-defined number due to the simple pole of $\zeta(1-z)$.

Before we proceed to analyze $z\in\{2,3,\ldots\}$ we recall from page 6 that $\Gamma(1-z)$ has simple poles for these z. For $z=2k, k\geq 1$, we find that $\sin\frac{\pi z}{2}$ has a simple zero, and as $\Gamma(1-z)$ has a simple pole we can use the same procedure as above. For $z = 2k+1, k\geq 1$, we face a different situation. As $\sin(\pi(k+\frac{1}{2}))$ no longer equals 0, we are forced to conclude that

$$\zeta(-2k) = 0, \quad k\geq 1. \tag{2.14}$$

These zeros are simple, cf. (2.12), as $\Gamma(1-2k)$ continues to have a simple pole.

We now fix z such that $-1<\Re(z)<0$ and consider the Fourier series of the first periodic Bernoulli polynomial according to (1.16). The trick of the proof is now to consider the integral from (2.10) as a limit for $p\to\infty$. After a bit of analysis, we see that this allows the order of summation and integration to be interchanged. By Lemma 9 we have

$$\left|2\sum_{k=1}^{n}\frac{\sin 2\pi kt}{2\pi k}\right| = \frac{1}{\pi}\left|\sum_{k=1}^{n}\frac{\sin 2\pi kt}{k}\right| \leq \frac{\pi+2}{\pi}, \quad t\in\mathbb{R},$$

and hence $1_{(0,p)}(t)2\sum_{k=1}^{n}\frac{\sin 2\pi kt}{2\pi k}$ is integrable. As z satisfies $-1<\Re(z)<0$, the function $1_{(0,p)}(t)t^{-z-1}$ will also be integrable for any p because the real part of t's exponent is always between 0 and 1. We find that

$$\left|1_{(0,p)}(t)\sum_{k=1}^{n}\frac{\sin 2\pi kt}{2\pi k\cdot t^{z+1}}\right| \leq 1_{(0,p)}(t)\frac{\pi+2}{\pi t^{\Re(z)+1}}.$$

We have found an integrable upper bound, and by Lebesgue's theorem B.2 we can interchange the order of summation and integration when inserting the Fourier series into (2.10):

$$\begin{aligned}\zeta(z) &= -z\lim_{p\to\infty}\int_0^p\frac{\overline{B}_1(t)}{t^{z+1}}dt = 2z\lim_{p\to\infty}\int_0^p\frac{1}{2\pi k}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin 2\pi kt}{t^{z+1}}dt\\ &= 2z\lim_{p\to\infty}\sum_{k=1}^{\infty}\int_0^p\frac{1}{2\pi k}\frac{\sin 2\pi kt}{t^{z+1}}dt,\end{aligned}$$

which we can rewrite as

$$\zeta(z) = 2z \lim_{p\to\infty} \sum_{k=1}^{\infty} \left(\int_0^\infty \frac{1}{2\pi k} \frac{\sin 2\pi k t}{t^{z+1}} dt - \int_p^\infty \frac{1}{2\pi k} \frac{\sin 2\pi k t}{t^{z+1}} dt \right).$$

Our strategy is now to show that the integral from p to ∞ approaches 0 sufficiently rapidly as p approaches ∞. This is not trivial, as t^{-z-1} is not integrable on (p, ∞). Let $x = \Re(z)$. Using integration by parts, we find

$$\begin{aligned} \left| \int_p^\infty \frac{\sin 2\pi k t}{t^{z+1}} dt \right| &\leq \left| \left[\frac{-\cos 2\pi k t}{2\pi k \cdot t^{z+1}} \right]_p^\infty \right| + \left| \int_p^\infty \frac{(z+1)\cos 2\pi k t}{2\pi k \cdot t^{z+2}} dt \right| \\ &\leq \frac{1}{2\pi k \cdot p^{x+1}} + \frac{|z+1|}{2\pi k} \int_p^\infty \frac{1}{t^{x+2}} dt = \frac{1}{2\pi k \cdot p^{x+1}} \left(1 + \frac{|z+1|}{x+1} \right), \quad x \in (-1, 0). \end{aligned}$$

Therefore the sum $\sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_p^\infty \frac{\sin(2\pi k t)}{t^{z+1}} dt$ is majorized by

$$\left(\sum_{k=1}^{\infty} \frac{1}{(2\pi k)^2} \right) \frac{1}{p^{x+1}} \left(1 + \frac{|z+1|}{x+1} \right) \to 0 \text{ as } p \to \infty.$$

Note that this could not be done without integrating by parts, as we need a higher power of t in the denominator, and that it is legal, provided we use a similar trick as in (2.8), and then let $q \to \infty$. With this result, the expression for $\zeta(z)$ now becomes

$$\begin{aligned} \zeta(z) &= 2z \sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_0^\infty \frac{\sin 2\pi k t}{t^{z+1}} dt = 2z \sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_0^\infty (2\pi k)^z \frac{\sin s}{s^{z+1}} ds \\ &= 2z \int_0^\infty \frac{\sin s}{s^{z+1}} ds \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^{1-z}} = 2z \int_0^\infty s^{-z-1} \sin s \, ds \, (2\pi)^{z-1} \zeta(1-z). \end{aligned}$$

Using the identity from (1.10) and the fact that $z\Gamma(-z) = -\Gamma(1-z)$ we finally get

$$\zeta(z) = -2z \sin \frac{\pi z}{2} \Gamma(-z) \, (2\pi)^{z-1} \zeta(1-z) = \frac{1}{\pi} (2\pi)^z \Gamma(1-z) \zeta(1-z) \sin \frac{\pi z}{2},$$

which shows (2.12). The difficult part of the proof is now done, but we still need to show (2.13). This turns out to involve only the formulae comprising the gamma function. Note the duplication formula (1.8) for complex argument (where x has been replaced by $z/2$):

$$\Gamma(z) = \frac{2^{z-1}}{\sqrt{\pi}} \Gamma\left(\frac{z+1}{2} \right) \Gamma\left(\frac{z}{2} \right)$$

Inserting $-z$ instead of z, and by multiplying both sides by z, we find

$$\Gamma(1-z) = -z\Gamma(-z) = \frac{2^{-z-1}}{\sqrt{\pi}} \Gamma\left(\frac{-z+1}{2} \right) \Gamma\left(\frac{-z}{2} \right) 2 \frac{-z}{2} = \frac{2^{-z}}{\sqrt{\pi}} \Gamma\left(\frac{1-z}{2} \right) \Gamma\left(\frac{2-z}{2} \right).$$

From Euler's formula (1.7), it follows that

$$\sin \frac{\pi z}{2} = \frac{\pi}{\Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{2-z}{2}\right)},$$

and inserting both identities into (2.12), we get

$$\zeta(z) = \frac{1}{\pi} (2\pi)^z \underbrace{\frac{2^{-z}}{\sqrt{\pi}} \Gamma\left(\frac{1-z}{2} \right) \Gamma\left(\frac{2-z}{2} \right)}_{\Gamma(1-z)} \zeta(1-z) \underbrace{\frac{\pi}{\Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{2-z}{2}\right)}}_{\sin \frac{\pi z}{2}},$$

and after rearranging, we get (2.13):

$$\zeta(z) \Gamma\left(\frac{z}{2} \right) \pi^{-\frac{z}{2}} = \zeta(1-z) \Gamma\left(\frac{1-z}{2} \right) \pi^{-\frac{1-z}{2}}.$$

□

Corollary 11 was proved in (2.14) as part of Theorem 10.

Corollary 11. *The zeta function has simple zeros at the even negative integers*

$$\zeta(-2n) = 0, \quad n \in \mathbb{N}.$$

These zeros are called the trivial zeros *of the zeta function.*

Before we go on, we want to sum up our results on the zeta function at integer values. Inserting $z = -2n + 1$ for $n \in \mathbb{N}$ into (2.12) and using (2.2), we find

$$\zeta(-2n+1) = \frac{1}{\pi}(2\pi)^{-2n+1}(2n-1)! \cdot (-1)^{n-1}\frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n} \cdot \sin \pi(-n + 1/2) = -\frac{B_{2n}}{2n}, \quad (2.15)$$

as $(-1)^{n-1} \sin \pi(-n + 1/2) = -1$ for all $n \in \mathbb{N}$. Recalling that the odd Bernoulli numbers except 1 all equal 0 and using that the zeta function has simple zeros, we obtain for $n \in \mathbb{N}$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

We now have an expression for all negative integers and the positive even integers, but what about the positive odd integers? $\zeta(3)$ is called Apéry's constant, named after Roger Apéry (1916-1994), who in 1977 proved that $\zeta(3)$ is irrational. It is known that infinitely many of the numbers $\zeta(2n+1)$ are irrational. Further, in 2001 Wadim Zudilin [9] proved a peculiar result: at least one of the numbers among $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational.

2.3 The critical strip and the Riemann hypothesis

The Riemann hypothesis—sometimes also referred to as the Riemann conjecture—is one of the most famous unsolved problems in mathematics. It is one of the Millennium Prize problems posed by the Clay Institute and it is among the famous Hilbert problems. David Hilbert said that, were he to sleep for 500 years, his first question upon awoking would be "did someone prove the Riemann hypothesis?". Unlike many other famous conjectures, many deep theorems are dependent on the truth of the Riemann hypothesis. So what is this hypothesis all about? The Riemann Hypothesis is a statement about the zeros of the (Riemann)zeta function, but before we state it let us first analyze the situation.

Let $f(z) := \zeta(z)\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}$. This is a meromorphic function in $\mathbb{C}\setminus\{0,1\}$. Why? The poles at $-2, -4, \ldots$ of $\Gamma\left(\frac{z}{2}\right)$ are "cancelled" by the zeros of the zeta function; cf. Corollary 11. The only remaining singularities are at $z = 0$, due to $\Gamma(z)$, and at $z = 1$ due to $\zeta(z)$. Further, according to the functional equation for the zeta function (Theorem 2.12), $f(z) = f(1-z)$, which means that f is symmetric with respect to $z = \frac{1}{2}$. As $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus 0$ and $\zeta(z) \neq 0$ for $\Re(z) > 1$, we conclude that $f(z) \neq 0$ whenever $1 < \Re(z)$ and by symmetry when $\Re(z) < 0$ as well. Hence, the zeta function cannot have any zeros outside the *critical strip* $0 \leq \Re(z) \leq 1$, except for the trivial zeros found in Corollary 11. The *critical line* are all the complex numbers z such that $\Re(z) = \frac{1}{2}$; see Figure 2.1.

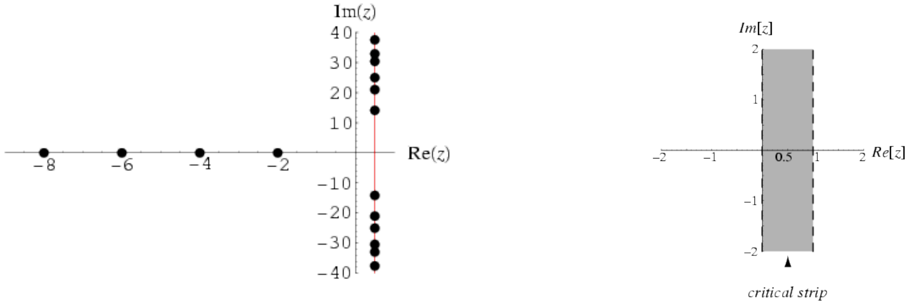

Figure 2.1: Zeros of the zeta function and the critical strip

Now it is time to formulate the Riemann hypothesis: Within the critical stripe, all the zeros of the zeta function lie on the critical line. Due to the symmetry of f around $z = \frac{1}{2}$

and the fact that $f(\overline{z}) = \overline{f(z)}$, one zero z_0 for the zeta function always produces four zeros—unless, of course, z_0 is invariant under one or both transformations. Note that the critical line is invariant under the transformation $z \mapsto 1 - z$ and, of course, $z \mapsto \overline{z}$.

Bernhard Riemann (1826–1866) published the paper containing his hypothesis in November 1859 called *Über die Anzahl der Primzahlen unter einer gegebenen Größe*.[3] Even though it was the only paper he ever published on number theory, it contained many important proofs and new notation, which you can read more about in [13].

Hardy's theorem was proved in 1915, and in 1989 Conrey [15] proved that more than 40% of the zeros of the zeta function lie on the critical line. ZetaGrid—a network of personal computers using their extra computing power when idle to find zeros of the zeta function—has verified the Riemann hypothesis for the first 100 billion zeros [16]. Odlyzko [14] provides a table of zeros.

2.4 Hardy's theorem

This section aims to prove Hardy's theorem, which states that $\zeta(z)$ has infinitely many zeros on the critical line. This does not imply that they all lie on the critical line, but nevertheless the result is interesting and a step towards a deeper level of understanding about the mechanics behind the Riemann hypothesis. Before proving Hardy's theorem, we have to introduce some new notation and state and prove a few minor theorems.

We start by introducing two spaces related to the Schwartz space $\mathcal{S}$, namely $\mathcal{S}_\infty$ and $\mathcal{S}_0$:

$$\mathcal{S}_\infty := \left\{ f \in C^\infty(\mathbb{R}_+) \mid \forall n, N \in \mathbb{N}_0 \exists C \, \forall x \geq 1 : \left| x^N \frac{d^n}{dx^n} f(x) \right| \leq C \right\}, \tag{2.16}$$

$$\mathcal{S}_0 := \left\{ f \in C^\infty(\mathbb{R}_+) \mid f(x^{-1}) \in \mathcal{S}_\infty \right\}. \tag{2.17}$$

Functions in $\mathcal{S}_\infty$ and $\mathcal{S}_0$ behave nicely as $x \to \infty$ and $x \to 0$ respectively. If $f \in \mathcal{S}_\infty$ then also $xf, f' \in \mathcal{S}_\infty$. Similarly for $f \in \mathcal{S}_0$, we find $x^{-1}f, f' \in \mathcal{S}_0$. If $f(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$, then $xf'(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$. Three functions of major importance are

$$G(x) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x^2}, \;\; H(x) := \frac{d}{dx} x^2 \frac{d}{dx} G(x) \text{ and } \xi(z) = \frac{1}{2} z(z-1)\zeta(z)\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}. \tag{2.18}$$

From the functional equation for the zeta function it is easily seen that $\xi(z) = \xi(1-z)$. As $\overline{\xi(z)} = \xi(\overline{z})$ it follows that $\xi\left(\frac{1}{2} + it\right)$ is a real and even function. We shall also consider the following sector

$$R := \{z \in \mathbb{C} \setminus \{0\} \mid \mathrm{Arg}(z) \in (-\pi/4, \pi/4)\}. \tag{2.19}$$

Lastly, we shall define the Fourier transform as a bijection of $\mathcal{S}$ onto itself.

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-ixt} dx, \quad t \in \mathbb{R}, \tag{2.20}$$

and with this definition the Fourier inversion formula reads

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt, \quad x \in \mathbb{R}. \tag{2.21}$$

We present a number of theorems whose conclusion will be Hardy's theorem (Theorem 17).

Theorem 12. *$\xi(z)$ is entire and*

$$2\xi(z) = \int_0^\infty x^{-z} H(x) dx \tag{2.22}$$

for $z \in \mathbb{C}$. Further, the following holds:

$$x^{1-a} H(x) = \frac{1}{\pi} \int_{\mathbb{R}} \xi(a+it) x^{it} dt \tag{2.23}$$

with $a, t \in \mathbb{R}$ and $x \in \mathbb{R}_+$.

[3] Usual English translation: On the number of primes less than a given magnitude.

Proof. The proof of (2.22) and that $\xi(z)$ is entire is omitted and can be found in Berg [2]. In order to prove (2.23), we start by rewriting the integral using the substitution $x = e^y$ and putting $z = a + it$, where $a, t \in \mathbb{R}$.

$$2\xi(z) = \int_0^\infty x^{-z} H(x)dx = \int_{\mathbb{R}} e^{-zy+y} H(e^y)dy = \int_{\mathbb{R}} e^{-ity} e^{y-ay} H(e^y)dy. \tag{2.24}$$

The idea of the proof is to use Fourier inversion on the last integral of (2.24), but before that can be done, we need to show that $e^{y-ay}H(e^y) \in \mathcal{S}$. We omit the fact that $x^{1-a}H(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$ of which the proof can be found in Berg [2]. We present a general result, which we can use on our function: Let $g : \mathbb{R} \to \mathbb{R}$ and assume

$$\exists f \in \mathcal{S}_0 \cap \mathcal{S}_\infty \forall x : g(x) = f(e^x), \tag{2.25}$$

then $g \in \mathcal{S}$. We need to show that $g \in C^\infty$ and that $x^N \frac{d^n}{dx^n} g(x)$ is bounded for all $N, n \in \mathbb{N}_0$. It is clear that g is in $C^\infty(\mathbb{R})$. Assuming that g satisfies (2.25), then $g'(x) = f'(e^x)e^x$ will also satisfy (2.25), as $f(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$ implies $xf'(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$. By induction, it follows that we only need to show that $x^N g(x)$ is bounded, as we have just seen that if g satisfies (2.25) then so do all of g's derivatives. Note that functions satisfying (2.25) are bounded, as $f \in \mathcal{S}_\infty$ ensures that a bound for $x \to \infty$ exists, and similarly, $f \in \mathcal{S}_0$ ensures we can find a bound for $x \to -\infty$. Further, $f(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$ implies $xf(x), x^{-1}f(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$ and hence $e^x g(x)$ and $e^{-x} g(x)$ must satisfy (2.25); in particular they must be bounded. As we can find sufficiently big k such that $|x^N| \le \max(e^x, e^{-x})$ for $|x| > k$, $x^N g(x)$ must also be bounded.

As $x^{1-a}H(x) \in \mathcal{S}_0 \cap \mathcal{S}_\infty$, we have now shown that $e^{y-ay}H(e^y) \in \mathcal{S}$, and can finally use Fourier inversion on (2.24):

$$e^{y-ay}H(e^y) = \frac{1}{2\pi} \int_{\mathbb{R}} 2\xi(a+it)e^{ity}dt$$

and once again letting $x = e^y$ we can simplify this to

$$x^{1-a}H(x) = \frac{1}{\pi} \int_{\mathbb{R}} \xi(a+it)x^{it}dt.$$

□

Theorem 13. *$G(z)$ and $H(z)$ are holomorphic in R, and for $z \in R$ the following equation holds:*

$$z^{\frac{1}{2}} H(z) = \frac{1}{\pi} \int_{\mathbb{R}} \xi\left(\frac{1}{2} + it\right) z^{it} dt \tag{2.26}$$

Proof. From (2.23) we know that the desired equation holds for $x \in \mathbb{R}_+$. If we can show that both sides are holomorphic in R, the equation follows. We only give a partial proof. It is quite technical to show that the right side i holomorphic, and we shall omit it; cf. Berg [2]. The fact that the left side is holomorphic is shown as follows.

For $z \in K \subset R$ where K is a compact set we find $|e^{-\pi n^2 z^2}| = e^{-\pi n^2((\Re(z))^2 - (\Im(z))^2)}$, and as K is compact we can find a $c > 0$ such that $(\Re(z))^2 - (\Im(z))^2) \ge c$. Hence the series for $G(x)$ will be dominated by $\sum_{n\in\mathbb{Z}} e^{-\pi n^2 c}$ in K. It follows that $G(x)$ and $H(x)$ are holomorphic in R, and in particular $G(x), H(x) \in C^\infty(\mathbb{R}_+)$. □

Theorem 14. *For $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ the following holds*

$$e^{i\frac{\theta}{2}} H(e^{i\theta}) = \sum_{n=0}^\infty c_{2n}\theta^{2n}, \quad c_{2n} = \frac{2}{\pi(2n)!} \int_0^\infty \xi\left(\frac{1}{2} + it\right) t^{2n} dt. \tag{2.27}$$

Proof. For $K = \{e^{i\theta} \mid |\theta| \le \frac{\pi}{4} - \varepsilon\}$ and remembering $\text{Log}z = \log|z| + i\,\text{Arg}z$, we find

$$z^{it} = e^{it\text{Log}z} = e^{-t\theta} = \sum_{n=0}^\infty \frac{(-\theta)^n}{n!} t^n,$$

as Arg$z = \theta$ when $z \in K$: that is when we can write z as $e^{i\theta}$. Inserting this into (2.26), we get

$$e^{i\frac{\theta}{2}} H(e^{i\theta}) = \frac{1}{\pi} \int_{\mathbb{R}} \xi \left(\frac{1}{2} + it \right) \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!} t^n dt$$

We want to interchange the order of integration and summation. To do this, we have to find an integrable upper bound, and then apply Lesbesgue's theorem C.2. As we have restricted ourselves to $z \in K$, it is clear that

$$\left| \sum_{n=0}^{m} \frac{(-\theta)^n}{n!} t^n \right| \leq e^{|t|(\frac{\pi}{4} - \varepsilon)},$$

for all $m \in \mathbb{N}$. As for $\xi\left(\frac{1}{2} + it\right)$, an upper bound can be found using Stirling's formula, which is done in Berg [2]. We can now interchange summation and integration to obtain

$$e^{i\frac{\theta}{2}} H(e^{i\theta}) = \sum_{n=0}^{\infty} c_n \theta^n, \quad \text{where } c_n = \frac{(-1)^n}{\pi n!} \int_{\mathbb{R}} \xi \left(\frac{1}{2} + it \right) t^n dt.$$

As $\xi(\frac{1}{2} + it)$ is a real and even function, $c_n = 0$ for odd n, which concludes our proof. □

Note that for θ to lie within $(-\frac{\pi}{4}, \frac{\pi}{4})$ means that $e^{i\theta}$ lies in R. We shall present the following Lemma 15 without a proof, which can be found in Berg [2].

Lemma 15. *For $z \in \mathbb{C} \setminus \{0\}$ we find*

$$G(z) = z^{-1} G(z^{-1}).$$

Theorem 16. *$G(z)$ and all its derivatives tend to 0 for $z \to i^{\frac{1}{2}} = e^{i\pi/4}$ along the unit circle. The same is true for $H(z)$ and all its derivatives.*

Proof. Recalling the definition of $G(z)$ from (2.18), it is easy to check the following identities:

$$\sum_{n \in \mathbb{Z}} (-1)^n e^{-\pi n^2 z^2} = 2G(2z) - G(z) \tag{2.28}$$

$$\sum_{n \in \mathbb{Z},\, n \text{ odd}} e^{-\pi n^2 z^2} = G(z) - G(2z). \tag{2.29}$$

Further, for $z \in R$, we also find that $(z-i)^{\frac{1}{2}} \in R$ and $(z-i)^{-\frac{1}{2}} \in R$, as $\Re(z^2 - i) = \Re(z^2) > 0$—which is equivalent to z's argument being in $(-\frac{\pi}{2}, \frac{\pi}{2})$—, and therefore the square root will be in R. Putting $\omega = (z^2 - i)^{-\frac{1}{2}}$ and using (2.28) we find

$$G(z) = \sum_{z \in \mathbb{Z}} e^{-\pi n^2 i} e^{-\pi n^2 (z^2 - i)} = \sum_{z \in \mathbb{Z}} (-1)^n e^{-\pi n^2 \omega^{-2}} = 2G\left(\frac{2}{\omega}\right) - G\left(\frac{1}{\omega}\right).$$

Applying Lemma 15 and the identity from (2.29),

$$G(z) = \omega G\left(\frac{\omega}{2}\right) - \omega G(\omega) = \omega \sum_{n \in \mathbb{Z},\, n \text{ odd}} e^{-\pi n^2 \left(\frac{\omega}{2}\right)^2},$$

which we will rewrite as

$$G(z) = 2 \sum_{n \in \mathbb{N},\, n \text{ odd}} \omega e^{-\frac{\pi}{4} n^2 \omega^2}.$$

We want to find the kth derivative of $G(z)$, and we therefore observe that $\frac{d\omega}{dz} = -z\omega^3$. Hence

$$G'(z) = 2 \sum_{n \in \mathbb{N},\, n \text{ odd}} \left(\omega e^{-\frac{\pi}{4} n^2 \omega^2} \right)' = \sum_{n \in \mathbb{N},\, n \text{ odd}} e^{-\frac{\pi}{4} n^2 \omega^2} \left(-2z\omega^3 + \pi n^2 z \omega^5 \right).$$

We are not interested in the exact form of $G^{(k)}(z)$, but rather we shall note as a generalization of the above calculation that the kth derivative is given as

$$G^{(k)}(z) = \sum_{n\in\mathbb{N},\, n \text{ odd}} e^{-\frac{\pi}{4}n^2\omega^2} P_k(z,\omega,n), \tag{2.30}$$

where $P_k(z,\omega,n)$ is a polynomial in z,ω and n. For each k, P_k can be thought of as a finite sum of powers of z multiplied by powers of ω and finally multiplied by powers of n. In other words,

$$P_k(z,\omega,n) = \sum_{j,m} z^j \omega^m P_{k,j,m}(n). \tag{2.31}$$

We want to find the limit for $z \to i^{\frac{1}{2}}$ and show that it is 0. Inserting (2.31) into (2.30), we see that it is enough to show that

$$\omega^m \sum_{n\in\mathbb{N},\, n \text{ odd}} P_{k,j,m}(n) e^{-\frac{\pi}{4}n^2\omega^2} \to 0$$

for each $m \in \mathbb{N}$, as z^j will tend to $i^{\frac{j}{2}}$. Then $G^{(k)}(z)$ would be a sum of finitely many zeros.

Consider $P_{k,j,m}(n)$. For each k, choose K such that K is strictly larger than the degrees of all the $P_{k,j,m}$ when j and m run through their finitely many indexes. Now $|P_{k,j,m}(n)| \le C_1 n^K$ for some constant C_1 and $n \in \mathbb{N}$. Fix $\lambda > 1$, then $\frac{n^K}{\lambda^n} \to 0$ for $n \to \infty$, and therefore we can find a constant C_2 such that $n^K \le C_2\lambda^n$. Recall that $\omega = (z^2 - i)^{-\frac{1}{2}}$, which means that letting $z \to i^{\frac{1}{2}}$ will have $\Re(\omega^2)$ tend to 0 and therefore $\lambda e^{-\frac{\pi}{4}\Re(\omega^2)} < 1$ after some point. We do not know what happens until then, but from that point on we can use the following estimate, which we will show tends to 0.

$$\begin{aligned}
\left|\omega^m \sum_{n\in\mathbb{N},\, n \text{ odd}} P_{k,j,m}(n) e^{-\frac{\pi}{4}n^2\omega^2}\right| &\le C_1|\omega|^m \sum_{n\in\mathbb{N},\, n \text{ a square}} n^K e^{-\frac{\pi}{4}n\Re(\omega^2)} \\
&\le C_1C_2|\omega|^m \sum_{n\in\mathbb{N}} \left(\lambda e^{-\frac{\pi}{4}\Re(\omega^2)}\right)^n \\
&= \frac{C_1C_2|\omega|^m \lambda e^{-\frac{\pi}{4}\Re(\omega^2)}}{1-\lambda e^{-\frac{\pi}{4}\Re(\omega^2)}}.
\end{aligned}$$

This last expression tends to 0 if $|\omega|^m e^{-\frac{\pi}{4}\Re(\omega^2)}$ tends to 0. We will want z to approach $i^{\frac{1}{2}}$ along the unit circle, and hence we start by writing $z = e^{i\theta} = \cos\theta + i\sin\theta$ where $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$. For z to approach $i^{\frac{1}{2}}$ along the unit circle corresponds to letting $\theta \nearrow \frac{\pi}{4}$. Now consider $\Re(\omega^2)$

$$\Re(\omega^2) = \Re\left(\frac{1}{z^2-i}\right) = \Re\left(\frac{\overline{z^2-i}}{(z^2-i)(\overline{z^2-i})}\right) = \frac{\Re(\overline{z}^2+i)}{|z^2-i|^2}. \tag{2.32}$$

As $z^2 = e^{i2\theta}$, we find that $|z^2-i|^2 = |\cos 2\theta + i(\sin 2\theta - 1)|^2 = 2 - 2\sin 2\theta$ and further that $\Re(\overline{z}^2 + i) = \cos 2\theta$. Inserting this into (2.32) we get

$$\Re(\omega^2) = \frac{\cos 2\theta}{2 - 2\sin 2\theta}.$$

Put $t = |z^2 - i| = \sqrt{2 - 2\sin 2\theta}$ in which case $\theta \nearrow \frac{\pi}{4}$ is equivalent to $t \searrow 0$. We find

$$\cos 2\theta = \sqrt{1-\sin^2 2\theta} = \sqrt{1-(1-t^2/2)^2} = t\sqrt{1-t^2/4}.$$

and further

$$\Re(\omega^2) = \frac{\sqrt{1-t^2/4}}{t} \to \infty \text{ for } t \searrow 0. \tag{2.33}$$

In conclusion, we see by inserting the expression found for $\Re(\omega^2)$ in (2.33) into $|\omega|^m e^{-\frac{\pi}{4}\Re(\omega^2)}$

$$\lim_{t\searrow 0}|\omega|^m e^{-\frac{\pi}{4}\frac{\sqrt{1-t^2/4}}{t}} = \frac{e^{-\frac{\pi}{4}\frac{\sqrt{1-t^2/4}}{t}}}{t^{m/2}} \to 0 \text{ for } t\searrow 0,$$

which concludes the proof of the theorem. □

Theorem 17 (Hardy's theorem)**.** *The function $t\mapsto \zeta(\frac{1}{2}+it)$ mapping from $\mathbb{R}$ to $\mathbb{R}$ has infinitely many zeros: that is $\zeta(z)$ has infinitely many zeros on the critical line.*

Proof. It is clear from (2.18) that if $\xi(z)$ has infinitely many zeros on the critical line so will $\zeta(z)$. We will show that this is true for $\xi(z)$ and proceed by contradiction. Assume that there are finitely many zeros: then we can choose $T>0$ such that $\xi(\frac{1}{2}+it)\neq 0$ for $t\geq T$. When $t\geq T$, $\xi(\frac{1}{2}+it)$ can no longer change its sign; let $s=\pm 1$ be the sign.

Multiplying both sides of (2.27) by s and rearranging, we get

$$s(2n)!\,\pi c_{2n} = 2\int_0^\infty s\,\xi\left(\frac{1}{2}+it\right)t^{2n}dt \geq 2\int_0^{T+2} s\,\xi\left(\frac{1}{2}+it\right)t^{2n}dt \tag{2.34}$$

as the integrand is positive due to s from T and onwards. For $t\in(0,T)$

$$s\,\xi\left(\frac{1}{2}+it\right)t^{2n}\leq\left|\xi\left(\frac{1}{2}+it\right)\right|T^{2n},$$

and therefore

$$\int_0^T s\,\xi\left(\frac{1}{2}+it\right)t^{2n}dt \geq -\int_0^T\left|\xi\left(\frac{1}{2}+it\right)\right|T^{2n}dt.$$

Using this fact in (2.34) produces

$$\begin{aligned}(2n)!\,\pi s c_{2n} &\geq 2\int_0^{T+2} s\,\xi\left(\frac{1}{2}+it\right)t^{2n}dt\\ &\geq 2\left(-\int_0^T\left|\xi\left(\frac{1}{2}+it\right)\right|T^{2n}dt+\int_{T+1}^{T+2}s\,\xi\left(\frac{1}{2}+it\right)(T+1)^{2n}dt\right)\\ &\geq -k_1T^{2n}+k_2(T+1)^{2n}=(T+1)^{2n}\left(k_2-k_1\left(\frac{T}{T+1}\right)^{2n}\right)\end{aligned}\tag{2.35}$$

where

$$k_1 = T\max_{t\in(0,T)}\left\{\left|\xi\left(\frac{1}{2}+it\right)\right|\right\}\quad\text{and}\quad k_2=\max_{t\in(T+1,T+2)}\left\{s\,\xi\left(\frac{1}{2}+it\right)\right\},$$

and hence both k_1 and k_2 are positive. It follows that we can find $n_0\in\mathbb{N}$ such that the last expression in (2.35) is positive for all $n\geq n_0$. We therefore have that $sc_{2n}>0$ for $n\geq n_0$. By Theorem 14 we find for $\theta\in(-\frac{\pi}{4},\frac{\pi}{4})$

$$\begin{aligned}s\frac{d^{2n_0}}{d\theta^{2n_0}}\left(e^{i\theta/2}H(e^{i\theta})\right) &= s\frac{d^{2n_0}}{d\theta^{2n_0}}\left(\sum_{n=0}^\infty c_{2n}\theta^{2n}\right)\\ &=\sum_{n=n_0}^\infty s\,c_{2n}\theta^{2n-2n_0}\frac{(2n)!}{(2n-2n_0+1)!} > s\,c_{2n_0}\end{aligned}\tag{2.36}$$

as we have just showed that all the terms of the sum after the n_0th term are positive. We are now ready to show our contradiction. We observe that

$$\frac{d^{2n_0}}{d\theta^{2n_0}}\left(e^{i\theta/2}H(e^{i\theta})\right)=\sum_{k=0}^{2n_0}\varphi_k(\theta)H^{(k)}(e^{i\theta})$$

where $\varphi_k(\theta)$ are all continuous. For $\theta\nearrow\frac{\pi}{4}$, we see that $H^{(k)}(e^{i\theta})\to 0$ by Theorem 16 which contradicts (2.36), and proves the theorem. □

2.5 Dirichlet series

Just as the zeta function considers $\sum_{n=1}^{\infty} n^{-z}$ as a function of z, the concept of *Dirichlet series* takes this abstraction one step further. Half planes will be of great importance in the following, and we define $\mathbb{H}_a := \{z \in \mathbb{C} \mid \Re(z) > a\}$.

Definition 18 (Dirichlet series)**.** *For any sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$, we define a* Dirichlet series *as*

$$D(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$$

for any $z \in \mathbb{C}$ and denote it by $D(z)$. If we need to specify the sequence $(a_n)_{n\in\mathbb{N}}$, we write $D_{a_n}(z)$.

We immediately note that the zeta function is a special case with $a_n = 1$ for all n. As with the zeta function, it is further clear that whether or not this series converges absolutely for a given z depends only on the real part of z. The following section focuses on two kinds of convergence and their respective domains. We define as follows:

Definition 19 (convergence of Dirichlet series)**.** *Let $\mathbb{C} \ni z = x + iy$. A Dirichlet series is said to be* absolutely convergent *if, for a given z,*

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^z} \right| = \sum_{n=1}^{\infty} \frac{|a_n|}{n^x} < \infty.$$

If this holds, the sum will be absolutely convergent in the closed half plane with $\Re(z) \geq x$. For a given Dirichlet series, we define the absolute convergence abscissa*, denoted by α, as*

$$\alpha := \inf\{\Re(z) | z \textit{ for which the Dirichlet series converges absolutely}\}.$$

The interpretation of α is that, for any complex number $z \in \mathbb{H}_\alpha$—that is to the right of the line with real part α—$D(z)$ will converge absolutely. With this interpretation, $\alpha = -\infty$ means that this is true for all of $\mathbb{C}$, and $\alpha = \infty$ tells us that $D(z)$ does not converge absolutely for any $z \in \mathbb{C}$. Note that it follows directly from Theorem C.1 that $D(z)$ is holomorphic in $\mathbb{H}_\alpha$.

It turns out that we can treat regular—that is not neccesarily absolute—convergence in the same way and split the complex plane in two. A priori this is not clear, as it could be that the sum would be convergent for real part x_0 but divergent for a z to the left as well as to the right of this line. This was first studied by Danish mathematician J.L.W.V Jensen (1859–1925) in 1884.

First, we shall prove a formula that resembles integration by parts for sums called Abel's summation formula. Let $(\alpha_k), (\beta_k) \in \mathbb{C}^{\mathbb{N}}$ and define $S_n = \sum_{k=1}^{n} \alpha_k$ with $S_0 := 0$. We shall show the following identity:

$$\sum_{k=n+1}^{n+p} \alpha_k \beta_k = S_{n+p}\beta_{n+p} - S_n\beta_{n+1} + \sum_{k=n+1}^{n+p-1} S_k(\beta_k - \beta_{k+1}). \tag{2.37}$$

And as $\alpha_k = S_k - S_{k-1}$,

$$\sum_{k=n+1}^{n+p} \alpha_k \beta_k = \sum_{k=n+1}^{n+p} (S_k - S_{k-1})\beta_k = \sum_{k=n+1}^{n+p} S_k\beta_k - \sum_{k=n}^{n+p-1} S_k\beta_{k+1},$$

where we have changed the index of summation in the last sum, remembering that $S_0 = 0$, and from this (2.37) follows.

Lemma 20. *Let α_n, β_n and S_n be as above. If*

1. *$|S_n| \leq A \in \mathbb{R}$ independent of n,*
2. *$\sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty$ and*
3. *$\beta_n \to 0$ for $n \to \infty$,*

then $\sum_{n=1}^{\infty} \alpha_n\beta_n$ is convergent and equal to the absolutely convergent series $\sum_{n=1}^{\infty} S_n(\beta_n - \beta_{n+1})$.

Proof. From (2.37) we see that

$$\begin{aligned}\left|\sum_{k=n+1}^{n+p}\alpha_k\beta_k\right| &\leq |S_{n+p}\beta_{n+p}|+|S_n\beta_{n+1}|+|S_k|\sum_{k=n+1}^{n+p-1}|\beta_k-\beta_{k+1}|\\ &\leq A\Big(|\beta_{n+p}|+|\beta_{n+1}|\Big)+A\sum_{k=n+1}^{n+p-1}|\beta_k-\beta_{k+1}|. \qquad (2.38)\end{aligned}$$

If we can show that for fixed N the above tends to 0 for any $n \geq N$ and for $p \to \infty$, we have established the convergence of the sum, as the "tail sum" will converge. Given $\varepsilon > 0$, assumption 3 shows that an N_1 exists, such that $|\beta_n| < \varepsilon$ for $n \geq N_1$, and by assumption 2 an N_2 exists, such that

$$\left|\sum_{k=n+1}^{n+p-1}(\beta_k-\beta_{k+1})\right| \leq \varepsilon$$

for $n \geq N_2$. For $n \geq N = \max\{N_1, N_2\}$ and for any $p \geq 1$, we therefore have

$$\left|\sum_{k=n+1}^{n+p}\alpha_k\beta_k\right| \leq 2A\varepsilon + A\varepsilon,$$

where A is independent of n by assumption 1. For $n = 0$, we in particular get

$$\sum_{k=1}^{p}\alpha_k\beta_k = S_p\beta_p + \sum_{k=1}^{p-1}S_k(\beta_k-\beta_{k+1}),$$

and on letting $p \to \infty$, we obtain

$$\sum_{k=1}^{\infty}\alpha_k\beta_k = \sum_{k=1}^{\infty}S_k(\beta_k-\beta_{k+1}),$$

as S_k is bounded and β_k converges to 0 □

We now allow $\beta_n(z)$ to be functions of z and formulate a similar lemma.

Lemma 21. *Let S_n and α_n be as above, let G be an open set, and let $\beta_n : G \to \mathbb{C}$ be a sequence of functions. If*

1. *$|S_n| \leq A \in \mathbb{R}$ independent of n,*
2. *$\sum_{n=1}^{\infty}|\beta_n(z)-\beta_{n+1}(z)|$ converges uniformly in G and*
3. *$\beta_n(z) \to 0$ for $n \to \infty$ and all $z \in G$,*

then $\sum_{n=1}^{\infty}\alpha_n\beta_n(z)$ converges uniformly in G and equals $\sum_{n=1}^{\infty}S_n(\beta_n(z)-\beta_{n+1}(z))$.

Proof. We will adapt the proof of Lemma 20: that is, we want to use the estimate in (2.38). S_n is once again bounded according to assumption 1, and assumption 2 shows that, given $\varepsilon > 0$, an N exists, such that

$$\left|\sum_{k=n+1}^{n+p-1}(\beta_k(z)-\beta_{k+1}(z))\right| \leq \sum_{k=n+1}^{n+p}|\beta_k(z)-\beta_{k+1}(z)| < \varepsilon \qquad (2.39)$$

for $n \geq N$, $p \geq 1$ and all $z \in G$. If we can show that $\beta_n(z) \to 0$ uniformly in G, we can use the estimate from (2.38). Consider

$$|\beta_n(z)-\beta_{n+p+1}(z)| = \left|\sum_{k=n}^{n+p}\beta_k(z)-\beta_{k+1}(z)\right| \leq \sum_{k=n}^{n+p}|\beta_k(z)-\beta_{k+1}(z)|.$$

But this is the same sum just estimated in (2.39), and on letting $p \to \infty$, we must have that

$$|\beta_n(z)| < \varepsilon \quad n \geq N, z \in G.$$

That is, $\beta_n(z) \to 0$ uniformly in G, and we have shown that $\sum_{k=1}^{\infty}\alpha_k\beta_k(z)$ is uniformly convergent in G, and it follows similarly to the proof of Lemma 20 that it equals $\sum_{k=1}^{\infty}S_k(\beta_k(z)-\beta_{k+1}(z))$. □

We can now prove the existence of a *convergence abscissa*—denoted γ—similar to the absolute convergence abscissa α introduced earlier.

Theorem 22. *If a Dirichlet series $D(z)$ is convergent for a $z_0 = x_0 + iy_0$, then $D(z)$ is uniformly convergent in any compact subset G of the half plane to the right of x_0: that is for $z \in \mathbb{H}_{x_0}$.*

Proof. Let $G \subseteq \mathbb{H}_{x_0}$ be such a compact subset, and let z_0 be such that $D(z_0)$ is convergent. Let

$$\alpha_n = \frac{a_n}{n^{z_0}} \text{ and } \beta_n(z) = \frac{1}{n^{z-z_0}}, \quad z = x + iy \in \mathbb{C}.$$

Hence $D(z) = \sum_{n=1}^{\infty} \alpha_n \beta_n(z)$. We now apply Lemma 21. Assumption 1 is fulfilled, as $D(z_0)$ is assumed to be convergent, and we observe that

$$|\beta_n(z)| = \frac{1}{n^{x-x_0}}$$

where the exponent is positive as $z \in \mathbb{H}_{x_0}$, and hence $\beta_n(z) \to 0$ for $n \to \infty$, which shows assumption 3. We turn to assumption 2 and use a trick to rewrite $\beta_n(z) - \beta_{n+1}(z)$ as

$$\beta_n(z) - \beta_{n+1}(z) = (z - z_0) \int_n^{n+1} \frac{1}{t^{z-z_0+1}} dt,$$

which we can estimate as

$$|\beta_n(z) - \beta_{n+1}(z)| = |z - z_0| \max_{t \in [n,n+1]} \frac{1}{t^{x-x_0+1}} = |z - z_0| \frac{1}{n^{x-x_0+1}}.$$

As G is compact and does not intersect the line $\Re(z) = x_0$, we find that

$$\max_{z \in G} |z - z_0| = M < \infty \text{ and } \min_{z \in G}(x - x_0) = d > 0,$$

and combining with the above, we see that

$$|\beta_n(z) - \beta_{n+1}(z)| \leq |z - z_0| \frac{1}{n^{x-x_0+1}} \leq M \frac{1}{n^{d+1}}.$$

As the exponent is strictly larger than 1, the infinite sum will converge, which shows assumption 2 of Lemma 21 and therefore $D(z)$ converges uniformly in G. □

From this, it follows that we can define γ as

$$\gamma := \inf\{\Re(z) | z \text{ for which the Dirichlet series converges}\},$$

and we see that $\gamma \leq \alpha$ for all Dirichlet series. From Theorem C.1 it follows that $D(z)$ is holomorphic in $\mathbb{H}_\gamma$ as it converges uniformly on compact sets, and these compact sets can fill up the open set of $\mathbb{H}_\gamma$. As we shall see, there is more to be said about the relationship between α and γ. Let $D(z)$ be convergent for $z_0 = x_0 + iy_0$. We then want to consider the absolute convergence of $D(z)$ with $z = x + iy$:

$$\left| \frac{a_n}{n^z} \right| = \frac{|a_n|}{n^{x_0}} \frac{1}{n^{x-x_0}}.$$

We know that

$$\frac{|a_n|}{n^{x_0}} \to 0 \quad \text{for } n \to \infty$$

as $D(z_0)$ converges. We can therefore find a constant A such that $\frac{|a_n|}{n^{x_0}} \leq A$ for all n, and we therefore find the following estimate

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^z} \right| \leq A \sum_{n=1}^{\infty} \frac{1}{n^{x-x_0}} = A\zeta(x - x_0),$$

which is finite if $x - x_0 > 1$. This means that $\gamma \leq \alpha \leq \gamma + 1$. Summing up, we find the following possibilities:

- $\gamma = \alpha = -\infty$
- $-\infty < \gamma \leq \alpha < \infty$ where $\gamma \leq \alpha < \gamma + 1$
- $\gamma = \alpha = \infty$.

This concludes our survey of the general theory for Dirichlet series, and we now turn to a specific case.

2.6 The Möbius function and the connection with the zeta function

First, let us define the Möbius function $\mu : \mathbb{N} \to \mathbb{R}$:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ different primes,} \\ 0, & \text{if a prime } p \text{ exists such that } p^2 | n. \end{cases}$$

Theorem 23. *The following holds:*

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}, \quad \Re(z) > 1, \tag{2.40}$$

where $\mu(n)$ denotes the Möbius function.

Proof. Let $p_k \in \mathbb{P}$ be the sequence of primes and $z \in \mathbb{C}$. The identity

$$\left(1 - \frac{1}{p_1^z}\right)\left(1 - \frac{1}{p_2^z}\right)\left(1 - \frac{1}{p_3^z}\right) = 1 - \sum_{k=1}^{3} \frac{1}{p_k^z} + \sum_{1 \le k_1 < k_2 \le 3} \frac{1}{(p_{k_1} p_{k_2})^z} - \frac{1}{(p_1 p_2 p_3)^3}$$

can be generalized to

$$\prod_{k=1}^{N} \left(1 - \frac{1}{p_k^z}\right) = 1 - \sum_{k=1}^{N} \frac{1}{p_k^z} + \sum_{1 \le k_1 < k_2 \le N} \frac{1}{(p_{k_1} p_{k_2})^z} - \sum_{1 \le k_1 < k_2 < k_3 \le N} \frac{1}{(p_{k_1} p_{k_2} p_{k_3})^z} + \ldots + (-1)^N \frac{1}{(p_1 \ldots p_N)^z}.$$

Here the first N primes are represented in the denominators to a power of either 0 or 1, and further, the sign in front of the sum depends on how many different primes there are in the denominator. It is 1 if there is an even number but –1 for an odd number of different primes. This gives us the connection to the Möbius function, and we find:

$$\prod_{k=1}^{N} \left(1 - \frac{1}{p_k^z}\right) = \sum_{n \in A_N} \frac{\mu(n)}{n^z}, \quad \text{where } A_N := \left\{p_1^{m_1} \cdots p_N^{m_N} | m_i \in \{0, 1\}\right\}.$$

We wish to include all powers of the first N primes, but as the Möbius function equals 0 if n is divisible by a square, we are simply adding a bunch of zeros and obtain

$$\prod_{k=1}^{N} \left(1 - \frac{1}{p_k^z}\right) = \sum_{n \in B_N} \frac{\mu(n)}{n^z} = D^{B_N}(z), \quad \text{where } B_N := \left\{p_1^{m_1} \cdots p_N^{m_N} | m_i \in \mathbb{N}_0\right\}.$$

$D^{B_N}(z)$ has absolute convergence abscissa $\alpha \le 1$ for all N, because the nth term of $D^{B_N}(z)$ is numerically smaller than the nth term of $\zeta(z)$. On the other hand, we know that $\sum \frac{1}{p_k} = \infty$ from (2.5), and therefore $\alpha \ge 1$, and hence $\alpha = 1$. On letting $N \to \infty$, we include all natural numbers in B_N, and as rearranging an absolute convergent series again gives us a convergent series, for $\Re(z) > 1$ we get

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$

using Euler's product (Theorem 7). □

We shall now express $\log \zeta(z)$ as a Dirichlet series, and first we define

$$c(n) = \begin{cases} \frac{1}{k}, & \text{if } n = p^k \text{ where } p \in \mathbb{P}, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Although we shall not prove the following theorem the proof can be found in Berg [2], and with the above definition we find the following

Theorem 24. *For $\Re(z) > 1$, a holomorphic branch of $\log\zeta(z)$ is given by*

$$\log\zeta(z) = \sum_{n=1}^{\infty} \frac{c(n)}{n^z} \tag{2.41}$$

This shall be useful in proving the following theorem, which uses a simple trigonometric inequality brilliantly

Theorem 25. *$\zeta(z) \neq 0$ for $\Re(z) = 1$; or, in other words, $\zeta(1 + it) \neq 0$ for $t \in \mathbb{R}$.*

Proof. By (2.41), for $x > 1$ and $y \in \mathbb{R}$ we have

$$\Re(\log\zeta(z)) = \Re\left(\sum_{n=1}^{\infty} \frac{c(n)}{n^z}\right) = \sum_{n=1}^{\infty} \frac{c(n)}{n^x}\cos(y\log n),$$

and in general for complex numbers the following holds:

$$\log z = \log|z| + i\arg(z),$$

from which it follows

$$\log|\zeta(x+iy)| = \Re(\log\zeta(x+iy)) = \sum_{n=1}^{\infty} \frac{c(n)}{n^x}\cos(y\log n)$$
$$\log|\zeta(x+2iy)| = \Re(\log\zeta(x+2iy)) = \sum_{n=1}^{\infty} \frac{c(n)}{n^x}\cos(2y\log n).$$

Using $\cos(2v) = \cos^2 v - \sin^2 v = 2\cos^2 v - 1$, which follows from the trigonometric addition formula, we see that

$$\cos(2v) + 4\cos v + 3 = 2(\cos v + 1)^2 \geq 0. \tag{2.42}$$

This leads us to the following inequality:

$$\log|\zeta(x+2iy)| + 4\log|\zeta(x+iy)| + 3\log\zeta(x) = \sum_{n=1}^{\infty} \frac{c(n)}{n^x}\left(\cos(2y\log n) + 4\cos(y\log n) + 3\right) \geq 0.$$

Taking the exponential on both sides, we get

$$\zeta(x)^3|\zeta(x+iy)|^4|\zeta(x+2iy)| \geq 1 \tag{2.43}$$

equivalent to

$$((x-1)\zeta(x))^3\left(\frac{|\zeta(x+iy)|}{x-1}\right)^4 (x-1)|\zeta(x+2iy)| \geq 1.$$

We now proceed by contradiction. Assume that a $y \neq 0$ exists such that $\zeta(1+iy) = 0$. Then

$$\frac{\zeta(x+iy)}{x-1} = \frac{\zeta(x+iy) - \zeta(1+iy)}{(x+iy) - (1+iy)} \to \zeta'(1+iy) \quad \text{for } x \to 1^+.$$

Further, from (2.9) it follows that $(x-1)\zeta(x) \to 1$ for $x \to 1^+$ and $\zeta(x+2iy) \to \zeta(1+2iy)$ for $x \to 1^+$, and inserting this into (2.43), we find

$$1^3 \cdot |\zeta'(1+iy)|^4 \cdot 0 \cdot |\zeta(1+2iy)| \geq 1$$

which gives us a contradiction. □

Note: We shall not prove the following but simply mention the connection of convergence of (2.40) and the Riemann hypothesis. The series in (2.40) is obviously not absolutely convergent for $z = 1$, but it can be shown to be convergent for z with $\Re(z) = 1$, and, in particular,

$$\frac{1}{\zeta(1)} = 0 = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}.$$

From the analysis of general Dirichlet series, we know that the convergence abscissa α must be in $[0, 1]$, but a glance at (2.40) reveals more about α than that. If $\alpha < \frac{1}{2}$, the series in (2.40) would extend to a holomorphic function in $\mathbb{H}_a$ with $a < \frac{1}{2}$, but this would include the critical line on which we know, from Hardy's theorem, that the zeta function has at least one zero and is therefore impossible. We conclude that $a \in [\frac{1}{2}, 1]$, but the exact value is not known. John Littlewood (1885–1977) showed in 1912 that $\alpha = \frac{1}{2}$ is equivalent to the Riemann hypothesis.

CHAPTER 3

The prime number theorem

Finally, we arrive at the second main result of this thesis: the acclaimed prime number theorem. It was originally based on empirical evidence in the form of tables and studied by Carl Friedrich Gauss (1777–1855) among others. Pafnuty Chebyshev (1821–1894) published two papers in 1848 and 1850 respectively, where he proved, that if

$$\frac{\pi(x)}{x/\log(x)}, \quad \pi(x) := \sum_{p \le x} 1 : \quad \text{that is } \pi(x) \text{ counts the number of primes less than } x$$

converges to a limit, this limit must be 1. One of the most significant papers published on the prime number theorem, *Über die Anzahl der Primzahlen unter einer gegebenen Größe* in 1859 by Riemann, introduces the idea of using complex analysis, the central idea in the proof that follows. The first proof was given independently by Jacques Hadamard (1865–1963) and Charles-Jean de la Vallée-Poussin (1866–1962) in 1896. The proof we give is essentially by Donald J. Newman (1930–2007) and based on the paper by Don Zagier (1951–) [3] published in 1997 on the 100th anniversary of the prime number theorem.

3.1 The Newman proof

Theorem 26 (Prime number theorem)**.** $\pi(x) \sim \frac{x}{\log x}$ *as* $x \to \infty$.
Here $f(x) \sim g(x)$ means that $\frac{f(x)}{g(x)} \to 1$ for $x \to \infty$.

Proof. We shall prove the theorem in five 5 steps. We first define two new functions

$$\Phi(z) = \sum_{p} \frac{\log p}{p^z}, \quad \theta(x) = \sum_{p \le x} \log p \quad (z \in \mathbb{C},\ x \in \mathbb{R}),$$

where p is a prime number. We also have to introduce some new notation: the *Big O notation*. $f(x) = \mathcal{O}(g(x))$ means that, on compact sets, f is bounded by a fixed multiple of $g(x)$.For instance, $e^x = 1 + x + \frac{x^2}{2} + \mathcal{O}(x^3)$.

1. $\theta(x) = \mathcal{O}(x)$.

Proof. For $n \in \mathbb{N}$, we have

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \ge \binom{2n}{n} \ge \prod_{n<p\le 2n} p = e^{\theta(2n)-\theta(n)},$$

and hence $\theta(2n) - \theta(n) \le 2n\log 2$. We now consider $x \in \mathbb{R}_+$ and want to estimate $\theta(2x) - \theta(x)$:

$$\theta(2x) - \theta(x) = \begin{cases} \theta(2n) - \theta(n) \le 2n\log 2, & x \in [n, n+\frac{1}{2}) \\ \theta(2n+1) - \theta(n) \le 2n\log 2 + \log(2n+1), & x \in [n+\frac{1}{2}, n+1) \\ 0, & 0 < x < 1 \end{cases} \tag{3.1}$$

The first line of the estimate is obvious. When x does not even get to $n + \frac{1}{2}$, neither $\theta(2n)$ nor $\theta(n)$ changes compared to $\theta(2x)$ and $\theta(x)$ respectively, and the original estimate holds. The second line is not difficult either. $\theta(n)$ will never change, but $\theta(2n)$ might. If it does, the worst thing that can happen is that $2n+1$ is a prime, in which case $\theta(2n+1) = \theta(2n) + \log(2n+1)$. Now we find $\theta(2n+1) - \theta(n) \le \theta(2n) + \log(2n+1) - \theta(n) \le 2n\log 2 + \log(2n+1)$ using our original estimate. The third line is trivial. Combining the estimates of (3.1), we find:

$$\theta(2x) - \theta(x) \le 2x\left(\log 2 + \frac{\log(2n+1)}{2n}\right), \quad x \in [n, n+1). \tag{3.2}$$

For each real $C > \log 2$, we can find $n_0 \in \mathbb{N}$ such that $\frac{\log(2n+1)}{2n} < C - \log 2$ for $n \ge n_0$, and using (3.2) we find

$$\theta(2x) - \theta(x) \le 2xC \quad \text{for } x \ge n_0,$$

which is equivalent to

$$\theta(x) - \theta(\frac{x}{2}) \le xC \quad \text{for } x \ge 2n_0. \tag{3.3}$$

We now find $r \in \mathbb{N}_0$ such that $x/2^r \ge 2n_0 > x/2^{r+1}$ for $x \ge 2n_0$, and we can write $\theta(x) - \theta(2n_0)$ as a telescoping sum

$$\theta(x) - \theta(2n_0) = \sum_{k=0}^{r}\left(\theta(\frac{x}{2^k}) - \theta(\frac{x}{2^{k+1}})\right) + \underbrace{\left(\theta(\frac{x}{2^{r+1}}) - \theta(2n_0)\right)}_{\text{last term}\le 0}. \tag{3.4}$$

For any $k \le r$, we have that $2n_0 \le x/2^r \le x/2^k$, and therefore $\theta(\frac{x}{2^k}) - \theta(\frac{x}{2^{k+1}}) \le \frac{x}{2^k}C$ by (3.3). Further, as θ is an non-decreasing function, the last term of (3.4) is less than or equal to 0. We therefore get

$$\theta(x) - \theta(2n_0) \le \sum_{k=0}^{r}\frac{x}{2^k}C \le Cx\sum_{k=0}^{\infty}\frac{1}{2^k} = 2Cx.$$

Hence $\theta(x)$—minus a constant—is less than or equal to a fixed multiple times x. This gives $\theta(x) = \mathcal{O}(x)$. □

Before the next step is proved, we recall that for a function f to be holomorphic in a closed set G means that for any $z_0 \in G$ we can find small disc $B(z_0, r)$ in which f is holomorphic.

2. $\Phi(z) - \frac{1}{z-1}$ *is holomorphic for* $\Re(z) \ge 1$ *or, equivalently,* $\Phi(z+1) - \frac{1}{z}$ *is holomorphic for* $\Re(z) \ge 0$.

Proof. We will use the Euler product from (2.3) to find the derivative of $\zeta(z)$. It is an infinite product, and differentiating, we get the sum of all the factors multiplied, where only the pth factor is being differentiated

$$\left(\prod_p f_p\right)' = f_1' f_2 f_3 \cdots + f_1 f_2' f_3 \cdots + f_1 f_2 f_3' \cdots + \ldots.$$

We divide by $\zeta(z)$ to remove all the factors in each term besides the ones that come from the differentation. Doing this, however, removes one factor too many, namely the pth, and we therefore have to multiply the pth term of the sum by $(1-p^{-s})$. We find

$$-\frac{\zeta(z)'}{\zeta(z)} = -\sum_{p\in\mathbb{P}}\left(\frac{1}{1-p^{-z}}\right)'(1-p^{-z}) = \sum_{p\in\mathbb{P}}\frac{p^{-z}\log p}{(1-p^{-z})^2}(1-p^{-z}) = -\sum_{p\in\mathbb{P}}\frac{p^{-z}\log p}{p^{-z}-1}$$
$$= \Phi(z) + \sum_{p\in\mathbb{P}}\frac{\log p}{p^z(p^z-1)}.$$

The final sum converges for $\Re(z) > \frac{1}{2}$, and as the zeta function is meromorphic in all of $\mathbb{C}$ with a pole at $z=1$, we must be able to extend $\Phi(z)$ to a meromorphic function with a pole at $z=1$ and poles at all the zeros of the zeta function. But Theorem 25 states that there are no zeros for $\zeta(z)$ on $\Re(z)=1$, so by subtracting $\frac{1}{z-1}$ we remove the only pole, and therefore it must be possible to extend $\Phi(z)-\frac{1}{z-1}$ to $\Re(z)\geq 1$. □

3 (analytical theorem)**.** *Let $f(t)$ be a bounded and locally integrable function for $t\geq 0$, and suppose that $g(z)=\int_0^\infty f(t)e^{-zt}dt$ where $\Re(z)>0$ extends holomorphically to $\Re(z)\geq 0$. Then $\int_0^\infty f(t)dt$ exists and equals $g(0)$.*

Proof. For $T>0$ define

$$g_T(z) = \int_0^T f(t)e^{-zt}dt.$$

As $f(t)$ is assumed to be bounded, this is clearly a holomorphic function for all z. We have assumed that $g(0)$ exists, and if we can show that $\lim_{T\to\infty} g_T(0) = g(0)$, this means that $\int_0^\infty f(t)dt$ exists if only as an improper integral.

Now consider a large R and the region $\{z\in\mathbb{C}\,|\,|z|\leq R, \Re(z)\geq -\delta\}$ of which C is the boundary. δ is constructed as follows: $g(z)$ is assumed to extend holomorphically to $\Re(z)\geq 0$ and therefore around any z on $\Re(z)=0$ we can find a small disc in which g is holomorphic. As the vertical line along the imaginary axis from $-R$ to R is compact, we can use Borel's covering theorem to find a finite number of these "holomorphic" discs that cover this line. If we choose δ to be small enough, we can make certain that g is holomorphic in and on C. We introduce the auxiliary function h given by

$$h(z) = (g(z)-g_T(z))e^{zT}\left(1+\frac{z^2}{R^2}\right).$$

which is the ingenious trick of the proof. By Cauchy's theorem, we find that

$$g(0)-g_T(0) = h(0) = \frac{1}{2\pi i}\int_C \frac{h(z)}{z}dz = \frac{1}{2\pi i}\int_C (g(z)-g_T(z))\,e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}. \tag{3.5}$$

The extra factor equals 1 for $z=0$ and therefore does not affect $g(0)-g_T(0)$. On the other hand, as we shall see, the extra factor allows us to find an estimate of the integrand. We now split C into $C_+ = C\cap\{\Re(z)>0\}$ and $C_- = C\cap\{\Re(z)<0\}$ and wish to estimate the latter integral in (3.5). First we look at C_+. We find that the integrand is bounded by $2B/R^2$ where $B=\max_{t\geq 0}|f(t)|$ is the bound of f

$$|g(z)-g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}dt\right| \leq B\int_T^\infty |e^{-zt}|dt = \frac{Be^{-\Re(z)T}}{\Re(z)}$$

as $\Re(z)>0$ in C_+. Further, as z lies on the boundary of C we have $z\overline{z}=|z|^2=R^2$

$$\left|e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\right| = e^{\Re(z)T}\left|\frac{1}{z}+\frac{z}{z\overline{z}}\right| = e^{\Re(z)T}\left|\frac{\overline{z}+z}{z\overline{z}}\right| = e^{\Re(z)T}\frac{2\Re(z)}{R^2}. \tag{3.6}$$

The contribution to the integral from (3.5) over C_+ is therefore the length of the semicircle times the maximum of the integrand over C_+. In conclusion, we find that the contribution to $g(0)-g_T(0)$ from C_+ is bounded by

$$\frac{1}{2\pi}\cdot\frac{2B}{R^2}\cdot R\pi = B/R.$$

We now consider C_-, and here we shall look at g and g_T separately. As g_T is entire, we can replace C_- by the "negative" semicircle: that is, by $C' = \{z \in \mathbb{C} \,|\, |z| = R, \Re(z) < 0\}$. For C', our estimate becomes

$$|g_T(z)| = \left| \int_0^T f(t)e^{-zt}dt \le B \int_{-\infty}^T |e^{-zt}|dt = \frac{Be^{-\Re(z)T}}{|\Re(z)|},\right.$$

as $\Re(z) < 0$. (3.6) holds equally well for $z \in C_-$, and we find the integrand to be bounded by $\frac{2\pi B}{R}$. Finally, the remaining integral

$$\int_{C_-} g(z)e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}dz$$

over C_- tends to 0 as $T \to \infty$, because e^{zT} is a factor of the integrand, which on compact sets converges uniformly to 0 as $\Re(z) < 0$. The rest of the integrand is independent of T, and the integral therefore converges to 0 by Lebesgue's theorem B.2. This shows us that $\limsup_{T\to\infty} |g(0) - g_T(0)| \le 2B/R$ but, as R was arbitrary, we conclude that $\lim_{T\to\infty} g_T(0) = g(0)$, which proves the theorem. □

4. $\int_1^\infty \frac{\theta(x)-x}{x^2}dx$ *is a convergent integral.*

Proof. We want to use the analytical theorem (step 3) on the functions

$$f(t) = e^{-t}\theta(e^t) - 1 \quad \text{and} \quad g(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}, \quad \Re(z) > 0.$$

From step 1, it follows that f is bounded and locally integrable. We need to show that $g(z) = \int_0^\infty f(t)e^{-zt}dt$, and to do so, we shall consider a Stieltjes integral, which we will not focus on in detail here. The idea behind the definition of this integral is the fact that it is possible to integrate with respect to a non-decreasing function, and in that case integration by parts still holds. For $\Re(z) > 1$, we then find[4]

$$\begin{aligned}\Phi(z) = \sum_{p\in\mathbb{P}} \frac{\log p}{p^z} &= \int_1^\infty \frac{d\theta(x)}{x^z} = \big[\theta(x)x^{-z}\big]_1^\infty + z\int_1^\infty \frac{\theta(x)}{x^{z+1}}dx \\ &= z\int_1^\infty \frac{\theta(x)}{x^{z+1}}dx = z\int_0^\infty e^{-zt}\theta(e^t)dt,\end{aligned}$$

where we have used $\lim_{x\to\infty}\theta(x)x^{-z} = 0$. From this and the fact that $\frac{1}{z} = \int_0^\infty e^{-zt}dt$, it follows that

$$g(z) = \int_0^\infty e^{-(z+1)t}\theta(e^t) - e^{-zt}dt = \int_0^\infty \big(e^{-t}\theta(e^t) - 1\big)e^{-zt}dt = \int_0^\infty f(t)e^{-zt}dt,$$

which was what we wanted to show. From step 2, it follows that $g(z)$ has a holomorphic extension to $\Re(z) \ge 0$, and the hypothesis of the analytical theorem is satisfied, and it follows that the following integral is convergent:

$$\int_0^\infty f(t)dt = \int_0^\infty \theta(e^t)e^{-t} - 1dt = \int_1^\infty (\theta(x)\frac{1}{x} - 1)\frac{1}{x}dx = \int_1^\infty \frac{\theta(x) - x}{x^2}dx$$

using the substitution $t = \log x$. □

5. $\theta(x) \sim x$.

Proof. We show that $\lim_{x\to\infty}\frac{\theta(x)}{x} = 1$. Assume $\lambda > 1$ exists such that $\theta(x) \ge \lambda x$ for arbitrary large x. We shall show that this leads to a contradiction. Since θ is a non-decreasing function, we find that

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2}dt \ge \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt = \int_x^{\lambda x} \frac{\lambda - t/x}{(t/x)^2}\frac{1}{x}dt = \int_1^\lambda \frac{\lambda - s}{s^2}ds > 0$$

[4]The second equality sign comes from the definition of the Stieltjes integral.

because the integrand is strictly positive on $(1, \lambda)$ using the substitution $s = t/x$. This is a positive constant independent of x, which contradicts

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2} dt \to 0 \quad \text{for } x \to \infty,$$

which is known to be true from 4 as $\int_1^\infty \frac{\theta(t)-t}{t^2} dx$ converges.

Similarly for $\lambda < 1$, assume that such a λ exists for which $\theta(x) \leq \lambda x$ for arbitrarily large x. This time, we find that

$$\int_{\lambda x}^{x} \frac{\theta(t) - t}{t^2} dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2} dt = \int_{\lambda}^{1} \frac{\lambda - s}{s^2} < 0$$

as our integrand this time is strictly negative on $(\lambda, 1)$. Once again, this number is independent of x, and by the same means as before, this contradicts the convergence of the integral $\int_1^\infty \frac{\theta(t)-t}{t^2} dx$. □

The prime number theorem follows easily from step 5, as all we need to show is

$$\lim_{x \to \infty} \frac{\pi(x) \log(x)}{x} = 1. \tag{3.7}$$

First we observe

$$\theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x,$$

and, from this, it follows that $\liminf_{x \to \infty} \frac{\pi(x) \log(x)}{x} \geq 1$, as for each x the quotient is greater than or equal to the corresponding quotient in a convergent sequence with limit 1. For any $1 > \varepsilon > 0$, we find

$$\frac{\theta(x)}{x} \geq \frac{1}{x} \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq \frac{1}{x} \sum_{x^{1-\varepsilon} \leq p \leq x} \log x^{1-\varepsilon}$$

where the first estimate holds as we sum over a smaller set, and the second because $x^{1-\varepsilon}$ is the smallest number in the set. Further, the number of elements in the sum must be less than $\pi(x) - x^{1-\varepsilon}$, and therefore

$$\frac{\theta(x)}{x} \geq (1 - \varepsilon) \frac{\log x}{x} (\pi(x) - x^{1-\varepsilon}) = (1 - \varepsilon) \left(\frac{\pi(x) \log x}{x} - \frac{\log x}{x^\varepsilon}) \right).$$

For the same reasons as before, and as the last term tends to 0 for $x \to \infty$, we obtain $\limsup_{x \to \infty} \frac{\pi(x) \log(x)}{x} \leq 1$. Hence the limit in (3.7) exists and equals 1, and we have proved the prime number theorem. □

Not all the arguments presented here are due to Newman. The clever technique used in step 1 was first used by Chebyshev, who used similar arguments to prove that the ratio of θ and x lies between 0.92 and 1.11. Newman proved the analytical theorem (step 3) and the use of it as shown in step 4 and 5 is also due to Newman.

CHAPTER 4

Perspective and conclusion

The two main results of this thesis are Hardy's theorem in relation to the Riemann hypothesis and the prime number theorem. Much more can be said about both, and the following is a brief perspective on the results and how they can be generalized.

One possible generalization of the prime number theorem is Dirichlet's theorem on primes in arithmetic progressions, which states that, for positive integers k and l being relatively prime, the arithmetic progression

$$k, k+l, k+2l, \ldots \tag{4.1}$$

contains infinitely many primes. This is parallel to Euclid's classical theorem stating that there are infinitely many primes. Similar to the generalization of this fact by the original prime number theorem, one can prove that

$$\pi_{k,l}(x) \sim \frac{1}{\varphi(l)} \frac{x}{\log x},$$

where $\pi_{k,l}$ denotes the number of primes in the arithmetic progression (4.1) and with $\varphi(l)$ being the Euler totient function.

Another generalization of the prime number theorem is to use the fundamental property of a prime p: if $p|ab$ then either $p|a$ or $p|b$. In algebraic ring theory, an ideal is a subset of a ring $\mathcal{R}$. Further, a prime ideal, P, is a proper ideal of $\mathcal{R}$ in which, for all $a, b \in \mathcal{R}$ with $ab \in \mathcal{P}$, either $a \in \mathcal{P}$ or $b \in \mathcal{P}$. Let $\pi_{\mathcal{R}}(X)$ denote the number of prime ideals in $\mathcal{R}$ with norm less than or equal to X. We then find

$$\pi_{\mathcal{R}}(X) \sim \frac{X}{\log X}.$$

This result is called the prime ideal theorem.

Finally, the Riemann hypothesis can be extended.[5] This would interplay perfectly with the prime ideal theorem. First, one needs to generalize the Riemann zeta function. Instead of summing over the integers, we can instead sum over the norms of non-zero ideals of a finite algebraic number field K. Such a function is called a Dedekind zeta function $\zeta_K(z)$. This can be extended analytically similar to how we extended the Riemann zeta function. The extended Riemann hypothesis states that, if $\zeta_K(z) = 0$ for $0 < \Re(z) < 1$, then $\Re(z) = \frac{1}{2}$. To prove the prime ideal theorem, one also needs to study Dedekind zeta functions.

[5] There is the generalized Riemann hypothesis concerning Dirichlet L-functions and there is the extended Riemann hypothesis concerning Dedekind zeta functions.

APPENDIX A

Convex functions

The term "convexity" applies to a vast number of different mathematical objects such as functions, sets and even relations which are used in such fields as economics. Common for all of these notions of "convexity" is the intuitive idea that the graphic representation of the object considered should be round and increasing. To formalize for functions:

Definition A.1 (convex function)**.** *$f : I \to \mathbb{R}$ is called* convex *if*

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) \tag{A.1}$$

for all $a < b \in I$ and $\lambda \in (0,1)$. It is called strictly *convex if equation (A.1) holds with $<$.*

Note that the right side is the chord between $f(a)$ and $f(b)$. The definition hence tells us that the actual value of the function for all $x \in [a,b]$ should be less than or equal to the values on this chord.

Definition A.2 (logarithmically convex function)**.** *$f : I \to \mathbb{R}$ is called* logarithmically convex *or simply* log-convex *if $\log(f(x))$ is convex. Or equivalently, if*

$$f(\lambda a + (1-\lambda)b) \leq f(a)^{\lambda} \cdot f(b)^{1-\lambda}, \quad \lambda \in (0,1),\ a < b. \tag{A.2}$$

We will usually show (A.2) rather than involving the logarithm, as this makes the calculations more transparent.

Example We will show that the beta function defined in (1.4) is log-convex when considered as a function of x.

$$\begin{aligned} B(\lambda a + (1-\lambda)b, y) &= \int_0^1 t^{\lambda a + (1-\lambda)b - 1}(1-t)^{y-1} dt \\ &= \int_0^1 \left(t^{a-1}(1-t)^{y-1}\right)^{\lambda} \left(t^{b-1}(1-t)^{y-1}\right)^{1-\lambda} dt \\ &\leq \left(\int_0^1 t^{a-1}(1-t)^{y-1} dt\right)^{\lambda} \left(\int_0^1 t^{b-1}(1-t)^{y-1} dt\right)^{1-\lambda} \\ &= B(a,y)^{\lambda} B(b,y)^{1-\lambda}, \end{aligned} \tag{A.3}$$

where we have used Hölder's theorem B.3 to establish the inequality.

Note that as log is a concave function, being log-convex implies being convex, and the product of two log-convex functions f and g is again log-convex, as

$$(f \cdot g)(a\lambda + (1-\lambda)b) \leq f(a)^{\lambda} f(b)^{1-\lambda} g(a)^{\lambda} g(b)^{1-\lambda} = (f \cdot g)(a)^{\lambda} (f \cdot g)(b)^{1-\lambda}.$$

APPENDIX B

Measure theory

Three major results from measure theory are used throughout this thesis. Consult Hansen [4] for details.

Theorem B.1 (Lebesgue's monotone convergence theorem)**.** *Let $(\mathcal{X}, \mathbb{E}, \mu)$ be a measure space and $f_n : \mathcal{X} \to [0, \infty]$ a sequence of $\mathcal{M}^+$-functions such that $f_n \nearrow f$ where f is the limit function, then $f \in \mathcal{M}^+$ and*

$$\int f_n d\mu \nearrow \int f d\mu.$$

Example As an application of Lebesgue's monotone convergence theorem, we want to show that

$$\lim_{x \to 1^+} \zeta(x) = \infty \quad \text{for } x \in \mathbb{R}.$$

Consider the sequence of functions for $x > 1$ given by $1_{]1,\infty[}(n)\frac{1}{n^x} = f_x(n) \nearrow f(n) = 1_{]1,\infty[}(n)\frac{1}{n}$. We shall consider the integral with respect to the counting measure τ. As all $f_x(n)$ are $\mathcal{M}^+$ functions, we can use Lebesgue's monotone convergence theorem to obtain

$$\zeta(x) = \int_1^\infty \frac{1}{n^x} d\tau \nearrow \int_1^\infty \frac{1}{n} d\tau = \infty. \tag{B.1}$$

Theorem B.2 (Lebesgue's dominated convergence theorem)**.** *Let $(\mathcal{X}, \mathbb{E}, \mu)$ be a measure space and $f_n : \mathcal{X} \to \mathbb{C}$ a sequence of $\mathcal{M}$ functions. If $h \in \mathcal{M}$ exists such that $|f_n(x)| \leq h(x)$ for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$, and with $\int h \, d\mu < \infty$ we shall call h an integrable upper bound for (f_n). Further,*

$$\int f_n \, d\mu \to \int f d\mu.$$

The above are two very important theorems from measure theory. [4] has more on measure theory. As part of showing that the gamma function is log-convex, we need the following theorem from function space theory. Recall that $\mathcal{L} = \{f | \int |f| \, dx < \infty\}$.

Theorem B.3 (Hölder)**.** *Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$\int |f(x)|^p dx < \infty \qquad \text{and} \qquad \int |g(x)|^q dx < \infty,$$

then $h = fg \in \mathcal{L}$ and

$$\int |h(x)| \, dx \leq \left(\int |f(x)|^p dx \right)^{1/p} \cdot \left(\int |g(x)|^q dx \right)^{1/q}.$$

Hölder's theorem is actually a bit more general. It gives a statement on any measurable function space instead of only the Lebesgue integrable functions, which is all that is needed in this thesis.

APPENDIX C

Complex analysis

This appendix summarizes two important theorems used in this thesis.

Theorem C..1. *Let $G \subseteq \mathbb{C}$ be open, and let (f_n) be a sequence of functions all of which are holomorphic on G. If f_n converges locally uniformly towards a function f, then f is holomorphic in G, and the sequence of (f_n') converges locally uniformly towards f' on G.*

Theorem C.2. *Let $G \subseteq \mathbb{C}$ be open and $f : G \times [a, b] \to \mathbb{C}$ have an integrable upper bound. That is, values of h exists such that*

$$f(z,t) \leq h(t) \text{ for all } z \in G, t \in [a,b] \text{ and } \int_a^b h(t)dt < \infty.$$

If $z \mapsto f(z,t)$ is holomorphic for each $t \in [a, b]$, then

$$F(z) = \int_a^b f(z,t)dt$$

is holomorphic in G with $F^{(n)}(z) = \int_a^b \frac{\partial^n f}{\partial z^n}(z,t)dt$.

NOTE: Usually we consider continuous f, which on a compact interval always has an integrable upper bound, namely itself.

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Index